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Source: *The Journal of Business*, Vol. 36, No. 4 (Oct., 1963), pp. 420-429
Published by: The University of Chicago Press
Stable URL: http://www.jstor.org/stable/2350971
Accessed: 08/12/2009 09:21

MANDELBROT AND THE STABLE PARETIAN HYPOTHESIS

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I. INTRODUCTION

THERE has long been a tradition among economists which holds that prices in speculative markets, such as grain and securities markets, behave very much like random walks.[1] The random walk theory is based on two assumptions: (1) price changes are independent random variables, and (2) the changes conform to some probability distribution. This paper will be concerned with the nature of the distribution of price changes rather than with the assumption of independence. Attention will be focused on an important new hypothesis concerning the form of the distribution which has recently been advanced by Benoit Mandelbrot. We shall see later that if Mandelbrot's hypothesis is upheld, it will radically revise our thinking concerning both the nature of speculative markets and the proper statistical tools to be used when dealing with speculative prices.

Prior to the work of Mandelbrot the usual assumption, which we shall henceforth call the Gaussian hypothesis, was that the distribution of price changes in a speculative series is approximately Gaussian or normal. In the best-known *theoretical* expositions of the Gaussian hypothesis, Bachelier[2] and Osborne[3] use arguments based on the central-limit theorem to support the assumption of normality. If the price changes from transaction to transaction are independent, identically distributed, random variables with finite variance, and if transactions are fairly uniformly spaced through time, the central-limit theorem leads us to believe that price changes across differencing intervals such as a day, a week, or a month will be normally distributed since they are simple sums of the changes from transaction to transaction. *Empirical* evidence in support of the Gaussian hypothesis has been offered by Kendall[4] and Moore.[5] Kendall found that weekly price changes for Chicago wheat and British common stocks were "approximately" normally distributed, and Moore reported similar results for the weekly changes in log price of a sample of stocks from the New York Stock Exchange.

Mandlebrot contends, however, that this past research has overemphasized agreements between empirical distributions of price changes and the normal distribution and has neglected certain departures from normality which are consistently observed. In particular, in most empirical work, Kendall's and Moore's

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[1] See, e.g., L. J. B. A. Bachelier, *Théorie de la speculation* (Paris: Gauthier-Villars, 1900); M. G. Kendall, "The Analysis of Economic Time Series, I: Prices," *Journal of the Royal Statistical Society*, Ser. A, XCVI (1953), 11–25; M. F. M. Osborne, "Brownian Motion in the Stock Market," *Operations Research*, VII (1959), 145–73; Harry V. Roberts, "Stock Market 'Patterns' and Financial Analysis: Methodological Suggestions," *Journal of Finance*, XIV (1959), 1–10; Paul H. Cootner, "Stock Prices: Random vs. Systematic Changes," *Industrial Management Review*, III (1962), 25–45; Arnold Moore, "A Statistical Analysis of Common Stock Prices" (unpublished Ph.D. dissertation, Graduate School of Business, University of Chicago, 1962); and S. S. Alexander, "Price Movements in Speculation Markets: Trends or Random Walks," *Industrial Management Review*, II (1961), 7–26.

[2] Bachelier, *op. cit.*

[3] Osborne, *op. cit.*

[4] Kendall, *op. cit.*

[5] Moore, *op. cit.*

included, it has been found that the extreme tails of empirical distributions are higher (i.e., contain more of the total probability) than those of the normal distribution. Mandelbrot feels that these departures from normality are sufficient to warrant a radically new approach to the theory of random walks in speculative prices. This new approach, which henceforth shall be called the stable Paretian hypothesis, makes two basic assertions: (1) the variances of the empirical distributions behave as if they were infinite, and (2) the empirical distributions conform best to the non-Gaussian members of a family of limiting distributions which Mandelbrot has called stable Paretian.[6]

The infinite variance assumption of the stable Paretian model has extreme implications. From a purely statistical standpoint, if the population variance of the distribution of first differences is infinite, the sample variance is probably a meaningless measure of dispersion. Moreover, if the variance is infinite, other statistical tools (e.g., least-squares regression) which are based on the assumption of finite variance will, at best, be considerably weakened and may in fact give very misleading answers. Since past research on speculative prices has usually been based on statistical tools which assume the existence of a finite variance, the value of much of this work may be doubtful if Mandelbrot's hypothesis is upheld by the data.

In the remainder of this paper we shall examine further the theoretical and empirical content of Mandelbrot's stable Paretian hypothesis. The first step will be to examine some of the important statistical properties of the stable Paretian distributions. The statistical properties will then be used to illustrate different types of conditions that could give rise to a stable Paretian market. After this the implications of the hypothesis for the theoretical and empirical work of the economist will be discussed. Finally, the state of the evidence concerning the empirical validity of the hypothesis will be examined.

II. THE STABLE PARETIAN DISTRIBUTIONS[7]

A. THE PARAMETERS OF STABLE PARETIAN DISTRIBUTIONS

The logarithm of the characteristic function for the stable Paretian family of distributions is[8]

$$\log f(t) = \log\int_{-\infty}^{\infty} \exp(iut)\, dP(\tilde{u} < u)$$

$$= i\delta t - \gamma\,|t|^{\alpha}[1 + i\beta(t/|t|)\tan(\alpha\pi/2)]\,.$$

The characteristic function tells us that stable Paretian distributions have four parameters, α, β, δ, and γ. The location parameter is δ, and if α is greater than one, δ is equal to the expectation or mean of the distribution. The scale parameter is γ, while the parameter β is

[6] To date Mandelbrot's most comprehensive published work in this area is "The Variation of Certain Speculative Prices," *Journal of Business*, October, 1963.

[7] The derivation of most of the important properties of stable Paretian distributions is due to P. Lévy, *Calcul des probabilitiés* (Paris: Gauthier Villars, 1925), 2d part, chap. vi. A rigorous and compact mathematical treatment of the statistical theory can be found in B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*, trans. K. L. Chung (Cambridge, Mass.: Addison-Wesley Press, 1954), chap. vii. A more comprehensive mathematical treatment can be found in Mandelbrot, *op. cit.* A descriptive treatment of the statistical theory is found in E. F. Fama, "The Distribution of the Daily First Differences of Stock Prices: A Test of Mandelbrot's Stable Paretian Hypothesis" (unpublished doctoral dissertation, University of Chicago, 1963).

[8] Lévy, *op. cit.*, p. 255. For an English-language derivation see Gnedenko and Kolmogorov, *op. cit.*, pp. 164–171.

an index of skewness which can take any value in the interval $-1 \leq \beta \leq 1$. When $\beta = 0$ the distribution is symmetric. When $\beta > 0$ the distribution is skewed right (i.e., has a long tail to the right), and the degree of right skewness increases in the interval $0 < \beta \leq 1$ as β approaches 1. Similarly, when $\beta < 0$ the distribution is skewed left, with the degree of left skewness increasing in the interval $-1 \leq \beta < 0$ as β approaches -1.

Of the four parameters of a stable Paretian distribution the characteristic exponent α is the most important for the purpose of comparing "the goodness of fit" of the Gaussian and stable Paretian hypotheses. The characteristic exponent α determines the height of, or total probability contained in, the extreme tails of the distribution, and can take any value in the interval $0 < \alpha \leq 2$. When $\alpha = 2$, the relevant stable Paretian distribution is the normal distribution.[9] When α is in the interval $0 < \alpha < 2$, the extreme tails of the stable Paretian distributions are higher than those of the normal distribution, with the total probability in the extreme tails increasing as α moves away from 2 and toward 0. The most important consequence of this is that the variance exists (i.e., is finite) only in the extreme case $\alpha = 2$. The mean, however, exists as long as $\alpha > 1$.[10]

Mandelbrot's stable Paretian hypothesis states that for distributions of price changes in speculative series α is in the interval $1 < \alpha < 2$, so that the distributions have means but their variances are infinite. The Gaussian hypothesis, on the other hand, states that α is exactly equal to 2.[11]

B. ESTIMATION OF α: THE ASYMPTOTIC LAW OF PARETO

Since the conflict between the stable Paretian and Gaussian hypotheses hinges essentially on the value of the characteristic exponent α, a choice between the hypotheses can be made, in theory, solely by estimating the true value of this parameter. Unfortunately, this is not a simple task. Explicit expressions for the densities of stable Paretian distributions are known for only three cases, the Gaussian ($\alpha = 2$), the Cauchy ($\alpha = 1$, $\beta = 0$), and the well-known coin-tossing case ($\alpha = \frac{1}{2}$, $\beta = 1$, $\delta = 0$, and $\gamma = 1$). Without density functions it is very difficult to develop and prove propositions concerning the sampling behavior of any estimators of α that may be used.[12]

The problem of estimation is not completely unsolvable, however. Although it is impossible to say anything about the sampling error of any given estimator of α, one can attempt to bracket the true value by using many different estimators. This is essentially the approach that I

[9] The logarithm of the characteristic function of a normal distribution is

$$\log f(t) = i\mu t - \frac{\sigma^2}{2} t^2 .$$

This is the logarithm of the characteristic function of a stable Paretian distribution with parameters $\alpha = 2$, $\delta = \mu$, and $\gamma = \sigma^2/2$.

[10] For a proof of these statements see Gnedenko and Kolmogorov, *op. cit.*, pp. 179–83.

[11] It is important to distinguish between the stable Paretian *distributions* and the stable Paretian *hypothesis*. Under *both* the stable Paretian and the Gaussian hypotheses it is assumed that the underlying distribution is stable Paretian. The conflict between the two hypotheses involves the value of the characteristic exponent α. The Gaussian hypothesis says that $\alpha = 2$, while the stable Paretian hypothesis says that α is strictly less than 2.

[12] Of course, these problems of estimation are not limited to the characteristic exponent α. The absence of explicit expressions for the density functions makes it very difficult to analyze the sampling behavior of estimators of all the parameters of stable Paretian distributions. The statistical intractability of these distributions is, at this point, probably the most important shortcoming of the stable Paretian hypothesis.

followed in my dissertation.[13] Three different techniques were used to estimate values of α for the daily first differences of log price for each individual stock of the Dow-Jones Industrial Average. Two of the estimation procedures, one based on certain properties of fractile ranges of stable Paretian variables and the other derived from the behavior of the sample variance, were introduced for the first time in the dissertation. An examination of these techniques would take us more deeply into the statistical theory of stable Paretian distributions than is warranted by the present paper. The third technique, double log graphing, is widely known, however, and will now be discussed in detail.

Lévy has shown that the tails of stable Paretian distributions for values of α less than 2 follow a weak or asymptotic form of the law of Pareto.[14] For distributions following the strong form of this law

$$P_r(\tilde{u} > u) = (u/V_1)^{-\alpha} \qquad u > 0\,, \quad (1)$$

and

$$P_r(\tilde{u} < u) = (|u|/V_2)^{-\alpha} \quad u < 0\,, \quad (2)$$

where $\tilde{u}$ is the random variable and the constants V_1 and V_2 are defined by

$$\beta = \frac{V_1^{\alpha} - V_2^{\alpha}}{V_1^{\alpha} + V_2^{\alpha}}.$$

β, of course, is the parameter for skewness discussed previously. The weak or asymptotic form of the law of Pareto is

$$P_r(\tilde{u} > u) \to (u/V_1)^{-\alpha} \text{ as } u \to \infty \qquad (3)$$

and

$$P_r(\tilde{u} < u) \to (|u|/V_2)^{-\alpha} \text{ as } u \to -\infty\,. \quad (4)$$

Taking logarithms of both sides of expressions (3) and (4), we have,

$$\log P_r(\tilde{u} > u) \to -\alpha(\log u - \log V_1)\,, \qquad u > 0 \qquad (5)$$

and

$$\log P_r(\tilde{u} < u) \to -\alpha(\log|u| - \log V_2)\,, \qquad u < 0\,. \qquad (6)$$

Expressions (5) and (6) imply that, if $P_r(\tilde{u} > u)$ and $P_r(\tilde{u} < u)$ are plotted against $|u|$ on double log paper, the two lines should become asymptotically straight and have slope that approaches $-\alpha$ as $|u|$ approaches infinity. Double log graphing, then, is one technique for estimating α.[15]

C. OTHER PROPERTIES OF STABLE PARETIAN DISTRIBUTIONS

The three most important properties of stable Paretian distributions are (1) the asymptotically Paretian nature of the extreme tail areas, (2) stability or invariance under addition, and (3) the fact that these distributions are the only possible limiting distributions for sums of independent, identically distributed, random variables. The asymptotic law of Pareto was discussed in the previous section. We shall now consider in detail the property of stability and the conditions under which sums of random variables follow stable Paretian limiting distributions.

1. *Stability or invariance under addition.*—By definition, a stable Paretian distribution is any distribution that is stable or invariant under addition. That is, the distribution of sums of independent, identically distributed, stable Paretian variables is itself stable Paretian

[13] *Op. cit.*, chap. iv.

[14] Lévy, *op. cit.*

[15] Unfortunately the simplicity of the double log graphing technique is, in some cases, more apparent than real. In particular, the technique is weak when the characteristic exponent is close to 2. For a discussion see Benoit Mandelbrot, "The Stable Paretian Income Distribution When the Apparent Exponent Is near Two," *International Economic Review*, IV (1963), 111–15, and also Fama, *op. cit.*, chap. iv.

and has the same form as the distribution of the individual summands. The phrase "has the same form" is, of course, an imprecise verbal expression of a precise mathematical property. A more rigorous definition of stability is given by the logarithm of the characteristic function of sums of independent, identically distributed, stable Paretian variables. The expression for this function is

$$n \log f(t) = i(n\delta)t - (n\gamma)\,|t|^{\alpha}\left[1 + i\beta \frac{t}{|t|}\left(\tan \frac{\alpha\pi}{2}\right)\right],$$

where n is the number of variables in the sum and $\log f(t)$ is the logarithm of the characteristic function of the individual summands. The above expression is exactly the same as the expression for $\log f(t)$, except that the parameters δ (location) and γ (scale) are multiplied by n. That is, the distribution of the sums is, except for origin and scale, exactly the same as the dsitribution of the individual summands. More simply, stability means that the values of the parameters α and β remain constant under addition.

The discussion above assumes that the individual, stable Paretian variables in the sum are independent and identically distributed. That is, the distribution of each individual summand has the same values of the four parameters α, β, δ, and γ. It will now be shown that stability still holds when the values of the location and scale parameters, δ and γ, are not the same for each individual variable in the sum. The logarithm of the characteristic function of sums of n such variable, each with different location and scale parameters, δ_j and γ_j, is

$$\sum_{j=1}^{n} \log f_j(t) = i\left(\sum_{j=1}^{n} \delta_j\right) t - \left(\sum_{j=1}^{n} \gamma_j\right)|t|^{\alpha}\left[1 + i\beta \frac{t}{|t|}\left(\tan \frac{\alpha\pi}{2}\right)\right].$$

This is the characteristic function of a stable Paretian distribution with parameters α and β, and with location and scale parameters equal, respectively, to the sums of the location and scale parameters of the distributions of the individual summands. That is, the sum of stable Paretian variables, where each variable has the same values of α and β but different location and scale parameters, is also stable Paretian with the same values of α and β.

The property of stability or invariance under addition is responsible for much of the appeal of stable Paretian distributions as descriptions of empirical distributions of price changes. The price change in a speculative series for any time interval can be regarded as the sum of the changes from transaction to transaction during the interval. If the changes between transactions are independent, identically distributed, stable Paretian variables, daily, weekly, and monthly changes will follow stable Paretian distributions of exactly the same form, except for origin and scale. For example, if the distribution of daily changes is normal with mean μ and variance σ^2, the distribution of weekly (or five-day) changes will also be normal with mean 5μ and variance $5\sigma^2$. It would be very convenient if the form of the distribution of price changes were independent of the differencing interval for which the changes were computed.

2. *Limiting distributions.*—It can be shown that stability or invariance under addition leads to a most important corollary property of stable Paretian distributions; they are the only possible limiting distributions for sums of independent, identically distributed, random variables.[16] It is well known that if such variables have finite variance the limiting

[16] For a proof see Gnedenko and Kolmogorov, *op. cit.*, pp. 162–63.

distribution for their sum will be the normal distribution. If the basic variables have infinite variance, however, and if their sums follow a limiting distribution, the limiting distribution must be stable Paretian with $0 < \alpha < 2$.

It has been proven independently by Gnedenko and Doeblin that in order for the limiting distribution of sums to be stable Paretian with characteristic exponent $\alpha (0 < \alpha < 2)$ it is necessary and sufficient that[17]

$$\frac{F(-u)}{1-F(u)} \rightarrow \frac{C_1}{C_2} \quad \text{as} \quad u \rightarrow \infty, \quad (7)$$

and for every constant $k > 0$,

$$\frac{1-F(u)+F(-u)}{1-F(ku)+F(-ku)} \rightarrow k^{\alpha} \text{ as } u \rightarrow \infty, \quad (8)$$

where F is the cumulative distribution function of the random variable $\tilde{u}$ and C_1 and C_2 are constants. Expressions (7) and (8) will henceforth be called the conditions of Doeblin and Gnedenko.

It is clear that any variable that is asymptotically Paretian (regardless of whether it is also stable) will satisfy these conditions. For example, consider a variable $\tilde{u}$ that is asymptotically Paretian but not stable. Then as $u \rightarrow \infty$

$$\frac{F(-u)}{1-F(u)} \rightarrow \left[\frac{(|-u|/V_2)}{(u/V_1)}\right]^{-\alpha} = \frac{V_2^{\alpha}}{V_1^{\alpha}},$$

and

$$\frac{1-F(u)+F(-u)}{1-F(ku)+F(-ku)} \rightarrow \frac{(u/V_1)^{-\alpha}+(|-u|/V_2)^{-\alpha}}{(ku/V_1)^{-\alpha}+(|-ku|/V_2)^{-\alpha}} = k^{\alpha},$$

and the conditions of Doeblin and Gnedenko are satisfied.

To the best of my knowledge nonstable, asymptotically Paretian variables are the only known variables of infinite variance that satisfy conditions (7) and (8). Thus they are the only known nonstable variables whose sums approach stable Paretian limiting distributions with characteristic exponents less than two.

[17] *Ibid.*, pp. 175–80.

III. THE ORIGIN OF A STABLE PARETIAN MARKET: SOME POSSIBILITIES

The price changes in a speculative series can be regarded as a result of the influx of new information into the market and of the re-evaluation of existing information. At any point in time there will be many items of information available. Thus price changes between transactions will reflect the effects of many different bits of information. The previous section suggests several ways in which these effects may combine to produce stable Paretian distributions for daily, weekly, and monthly price changes.

In the simplest case the price changes implied by individual bits of information may themselves follow stable Paretian distributions with constant values for the parameters α and β, but possibly different values for the location and scale parameters, δ and γ. If the effects of individual bits of information combine in a simple, additive fashion, then by the property of stability the price changes from transaction to transaction will also be stable Paretian with the same values of the parameters α and β. Since the price changes for intervals such as a day, week, or month are the simple sums of the changes from transaction to transaction, the changes for these intervals will also be stable Paretian with the same values of the parameters α and β.

Now suppose the price changes implied by individual items of information are asymptotically Paretian but not stable. This means that the necessary and sufficient conditions of Doeblin and Gnedenko will be satisfied. Thus if the effects of individual bits of information combine in a simple, additive fashion, and if there are very many bits of information involved in a transaction, the distribution of price changes between transactions will be stable Paretian. It may happen, however, that there are not enough bits of information involved in individual transactions to insure that the limiting stable Paretian distribution is closely achieved by the distribution of changes from transaction to transaction. In this case as long as there are many transactions per day, week, or month, the distributions of price changes for these differencing intervals will be stable Paretian with the same values of the parameters α and β.

Mandelbrot has shown that these results can be generalized even further.[18] As long as the effects of individual bits of information are asymptotically Paretian, various types of complicated combinations of these effects will also be asymptotically Paretian. For example, although there are many bits of information in the market at any given time, the price change for individual transactions may depend solely on what the transactors regard as the largest or most important piece of information. Mandelbrot has shown that, if the effects of individual items of information are asymptotically Paretian with exponent α, the distribution of the largest effect will also be asymptotically Paretian with the same exponent α. Thus the distribution of changes between transactions will be asymptotically Paretian, and the conditions of Doeblin and Gnedenko will be satisfied. If there are very many transactions in a day, week, or month, the distributions of price changes for these differencing intervals will be stable Paretian with the same value of the characteristic exponent α.

In sum, so long as the effects of individual bits of information combine in a way which makes the price changes from transaction to transaction asymptotically *Paretian* with exponent α, then according to the conditions of Doeblin and Gnedenko the price changes for longer differencing intervals will be *stable* Paretian with the same value of α. According to our best knowledge at this time, however, it is necessary that the distribution of the price changes implied by individual bits of information be at least *asymptotically Paretian* (but not necessarily stable) if the distributions of changes for longer time periods are to have *stable Paretian* limits.

IV. IMPORTANCE OF THE STABLE PARETIAN HYPOTHESIS

The stable Paretian hypothesis has many important implications. First of all, if we retrace the reasoning of the previous section, we see that the hypothesis implies that there are a larger number of abrupt changes in the economic variables that determine equilibrium prices in speculative markets than would be the case under a Gaussian hypothesis. If the distributions of daily, weekly, and monthly price changes in a speculative series are *stable* Paretian with $0 < \alpha < 2$, the distribution of changes between transactions must, at very least, be asymptotically Paretian. Changes between transactions are themselves the result of the combination of the effects of many different bits of information. New information, in turn, should ultimately reflect changes in the underlying economic conditions that determine equilibrium prices in speculative markets.

[18] Mandelbrot, "New Methods in Statistical Economics," *Journal of Political Economy*, October, 1963.

Thus, following this line of reasoning, the underlying economic conditions must themselves have an asymptotically Paretian character and are therefore subject to a larger number of abrupt changes than would be the case if distributions of price changes in speculative markets conformed to the Gaussian hypothesis.

The fact that there are a large number of abrupt changes in a stable Paretian market means, of course, that such a market is inherently more risky for the speculator or investor than a Gaussian market. The variability of a given expected yield is higher in a stable Paretian market than it would be in a Gaussian market, and the probability of large losses is greater.

Moreover, in a stable Paretian market speculators cannot usually protect themselves from large losses by means of such devices as "stop-loss" orders. In a Gaussian market if the price change across a long period of time is very large, chances are the total change will be the result of a large number of very small changes. In a market that is stable Paretian with $\alpha < 2$, however, a large price change across a long interval will more than likely be the result of a few very large changes that took place during smaller subintervals.[19] This means that if the price level is going to fall very much, the total decline will probably be accomplished very rapidly, so that it may be impossible to carry out many "stop-loss" orders at intermediate prices.[20]

[19] For a proof of these statements see Donald Darling, "The Influence of the Maximum Term in the Addition of Independent Random Variables," *Transactions of the American Mathematical Society*, LXXIII (1952), 95–107, or D. Z. Anov and A. A. Bobnov, "The Extreme Members of Samples and Their Role in the Sum of Independent Variables," *Theory of Probability and Its Applications*, V (1960), 415–35.

[20] Mandelbrot, "The Variation of Certain Speculative Prices," *op. cit.*

The inherent riskiness of a stable Paretian market may account for certain types of investment behavior which are difficult to explain under the hypothesis of a Gaussian market. For example, it may *partially* explain why many people avoid speculative markets altogether, even though at times the expected gains from entering these markets may be quite large. It may also partially explain why some people who are active in these markets hold a larger proportion of their assets in less speculative, liquid reserves than would seem to be necessary under a Gaussian hypothesis.

Finally, the stable Paretian hypothesis has important implications for data analysis. As mentioned earlier, when $\alpha < 2$ the variance of the underlying stable Paretian distribution is infinite, so that the sample variance is an inappropriate measure of variability. Moreover, other statistical concepts, such as least-squares regression, which are based on the assumption of finite variance are also either inappropriate or considerably weakened.

The absence of a finite variance does *not* mean, however, that we are helpless in describing the variability of stable Paretian variables. As long as the characteristic exponent α is greater than 1, estimators which involve only first powers of the stable Paretian variable have finite expectation. This means that concepts of variability, such as fractile ranges and the absolute mean deviation, which do involve only first powers, have finite expectation and thus are more appropriate measures of variability for these distributions than the variance.[21]

[21] A fractile range shows the range of values of the random variable that fall within given fractiles of its distribution. For example, the interquartile range shows the range of values of the random variable

V. THE STATE OF THE EVIDENCE

The stable Paretian hypothesis has far-reaching implications. The nature of the hypothesis is such, however, that its acceptability must ultimately depend on its empirical content rather than on its intuitive appeal. The empirical evidence up to this point *has* tended to support the hypothesis, but the number of series tested has not been large enough to warrant the conclusion that further tests are unnecessary.

For commodity markets the most impressive single piece of evidence is a direct test of the infinite variance hypothesis for the case of cotton prices. Mandelbrot computed the sample second moments of the daily first differences of the logs of cotton prices for increasing samples of from 1 to 1,300 observations. He found that as the sample size is increased the sample moment does not settle down to any limiting value but rather continues to vary in absolutely erratic fashion, precisely as would be expected under the stable Paretian hypothesis.[22]

Mandelbrot's other tests in defense of the stable Paretian hypothesis are based primarily on the double log graphing procedure mentioned earlier. If the distribution of the random variable $\tilde{u}$ is stable Paretian with $\alpha < 2$, the graphs of log $P_r(\tilde{u} < u)$, u negative, and log $P_r(\tilde{u} > u)$, u positive, against log $|u|$ should be curves that become asymptotically straight with slope $-\alpha$. The graphs for the same cotton price data seemed to support the hypothesis that α is less than 2. The empirical value of α appeared to be about 1.7.

Finally, in my dissertation the stable Paretian hypothesis has been tested for the daily first differences of log price of each of the thirty stocks in the Dow-Jones Industrial Average. Simple frequency distributions and normal probability graphs were used to examine the tails of the empirical distributions for each stock. In *every* case the empirical distributions were long-tailed, that is, they contained many more observations in their extreme tail areas than would be expected under a hypothesis of normality. In addition to these tests three different procedures were used to estimate values of the characteristic exponent α for each of the thirty stocks. The estimates produced empirical values of α *consistently* less than 2. The conclusion of the dissertation is that for the important case of stock prices the stable Paretian hypothesis is more consistent with the data than the Gaussian hypothesis.

VI. CONCLUSION

In sum, the stable Paretian hypothesis has only been directly tested on a limited number of different types of speculative price series. It should be emphasized, however, that every direct test on unprocessed and unsmoothed price data has found the type of behavior that would be predicted by the hypothesis. Before the hypothesis can be accepted as a general model for speculative prices, however, the basis of testing must be broadened to include other speculative series.

Moreover, the acceptability of the stable Paretian hypothesis will be improved not only by further empirical documentation of its applicability but also by making the distributions themselves more

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that fall within the 0.25 and 0.75 fractiles of the distribution.

The absolute mean deviation is defined as

$$|D| = \sum_{i=1}^{N} \frac{|X_i - \bar{X}|}{N},$$

where N is the total sample size.

[22] Mandelbrot, "The Variation of Certain Speculative Prices," *op. cit.*

tractable from a statistical point of view. At the moment very little is known about the sampling behavior of procedures for estimating the parameters of these distributions. Unfortunately, as mentioned earlier, rigorous, analytical sampling theory will be difficult to develop as long as explicit expressions for the density functions are not known. However, pending the discovery of such expressions, Monte Carlo techniques could be used to learn some of the properties of various procedures for estimating the parameters.

Mandelbrot's stable Paretian hypothesis has focused attention on a long-neglected but important class of statistical distributions. It has been demonstrated that among speculative series the first differences of the logarithms of stock and cotton prices seem to conform to these distributions. The next step must be both to test the stable Paretian hypothesis on a broader range of speculative series and to develop more adequate statistical tools for dealing with stable Paretian distributions.