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Liapunov–Razumikhin Conditions for Asymptotic Stability in Functional Differential Equations of Volterra Type

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In [1], B.S. Razumikhin gave conditions for the stability and asymptotic stability of the zero state of systems of ordinary differential equations involving a fixed finite time delay. His conditions make use of Liapunov functions defined on the finite-dimensional state space rather than on the space of functions continuous on the interval of delay; for methods involving functions of the latter type, cf., for example, Yoshizawa [2].

In a recent paper [3], the author has used Liapunov functions of Razumikhin type to give conditions sufficient for the stability of the zero state of a system of ordinary differential equations involving an interval of delay which becomes unbounded as $t \to +\infty$; an example of such a system is an integrodifferential equation of Volterra type:

$$\dot{x}(t) = G(t, x(t)) + \int_0^t K(t, s, x(s))\, ds, \tag{1.1}$$

here and henceforth, $\dot{x}(t) = dx/dt$, and x, G, and K denote functions with values in R^n, n-dimensional Euclidean space.

An example given in [3] and in the appendix of this paper shows that the usual modifications which in the bounded delay case yield conditions sufficient for asymptotic stability of the zero state do not work for systems like (1.1) with an unbounded interval of delay. It is the purpose of this paper to give conditions in terms of Liapunov–Razumikhin functions sufficient for asymptotic stability. The main idea of these conditions is that the effect of states near $t = 0$ on the rate of change (the derivative of the state at $t > 0$ should decrease rapidly as t increases.

We use the following notation and definitions: $R = R^1$, with R^n as previously defined; for $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ in R^n, we define $x \cdot y = \sum_{j=1}^n x_j y_j$, and $|x| = (x \cdot x)^{1/2}$. For each $t > 0$, we denote by S_t the set $\{x_t(\cdot)\}$ of functions continuous on $[0, t]$ to R^n. We observe that if $x(s)$ is a function continuous on $[0, T)$ to R^n, $T > 0$, and if $t \in (0, T)$, then $x(s)$,

$s \in [0, t]$ is a member of S_t : here and henceforth, $T = +\infty$ is allowed. We call such an element $x_t(\cdot)$ of S_t a segment of $x(s)$, $s \in [0, T)$.

For fixed $t > 0$, let $F(t, (\cdot))$ denote a function on S_t to R^n. We assume that if $x_t(\cdot)$ is the segment of a continuous function $x(s)$, $s \in [0, T)$, then $F(t, x_t(\cdot))$ is continuous on $[0, T)$. By a solution of

$$\dot{x}(t) = F(t, x_t(\cdot)), \tag{1}$$

we mean a continuously differentiable function $x(s)$, $s \in [0, T)$, such that (1) holds for $t \in (0, T)$. We shall assume that if $T < \infty$ and $x(s)$ is a solution on $[0, T)$ which cannot be continued in the usual way to the right of T, then $x(s)$ cannot be bounded on $[0, T)$. This is essentially hypothesis (H_1) in [3]; for conditions on G and K in (1.1) sufficient that (H_1) hold for this system, cf. [4], Corollary 4, p. 98.

We denote by $x(t, x_0)$ any solution of (1) such that $x(0, x_0) = x_0$. We use the usual Liapunov definitions of stability and asymptotic stability; for the former, cf. [3], Definition 1, and for the latter:

DEFINITION. $x = 0$ is said to be asymptotically stable for (1) if it is stable and if there exists a $r_0 > 0$, such that $|x_0| < r_0$ implies $x(t, x_0) \to 0$ as $t \to +\infty$.

In what follows, $V(t, x)$ denotes a function continuous on $R \times R^n$ to R. We assume that:

(a) There exist real-valued functions $u(s)$ and $v(s)$, continuous and increasing for $s \geqslant 0$ such that $u(0) = v(0) = 0$, and

$$u(|x|) \leqslant V(t, x) \leqslant v(|x|) \qquad \text{for} \quad t \geqslant 0, \quad x \in R^n; \tag{2}$$

(b) Let $f(s)$ be a function continuous for $s \geqslant 0$ such that $f(s) > s$ for $s > 0$. Let $x(t)$ be a solution of (1) on $[0, T)$ $T \leqslant \infty$. Then there exists a number $r > 0$, and a function $w(s)$, continuous for $s \geqslant 0$, $w(0) = 0$, and $w(s) > 0$ for $s > 0$, such that the condition

(i) $V(s, x(s)) < f(V(t, x(t)))$ for $s \in [t_0, t]$, $t > 0$, where $t_0 = \max[0, t - r]$, implies

(ii) $\dot{V}(t, x(t)) \leqslant -w(|x(t)|)$;

here $w(s)$ and r may depend on the solution $x(t)$, as well as on V and f, and

$$\dot{V}(t, x(t)) = \limsup_{h \to 0-} [V(t + h, x(t + h)) - V(t, x(t))]/h.$$

THEOREM 1. *Let there exist functions V and f satisfying the conditions* (a) *and* (b) *above. Then for each solution $x(t)$ bounded on* $[0, \infty)$, $x(t) \to 0$ *as* $t \to +\infty$.

COROLLARY 1. *If, in addition to the hypotheses of Theorem* 1, $x = 0$ *is stable for* (1), *then it is asymptotically stable.*

COROLLARY 2. *If, in addition to* (a), (b) *holds for any solution* (*not necessarily bounded on* $[0, \infty)$), *then* $x = 0$ *is asymptotically stable for* (1).

Corollary 1 follows easily from the theorem: all we need observe is that as a consequence of the stability of $x = 0$ for (1), there exists a $r_0 > 0$ such that $|x(t, x_0)| < 1$ for $|x_0| < r_0$.

To prove Corollary 2, we observe if condition (b) holds for any solution of (1), then this condition implies hypothesis (H_2) of Theorem 1 in [3] which states that for f, V, and $\dot{V}$ as above, if $x(s)$ is any solution of (1) which exists on $[0, t]$, $t > 0$, for which $f(V(t, x(t))) > V(s, x(s))$ for $s \in [0, t]$, then

$$\dot{V}(t, x(t)) \leqslant 0.$$

Since the conclusion of Theorem 1 in [3] is that $x = 0$ is stable for (1), Corollary 1, and hence Corollary 2 follows.

Proof of Theorem 1. The proof is essentially the same as the one for the case where the time delay is fixed and finite; we give it here for the sake of completeness.

Let $x(t)$ be a solution of (1) bounded on $[0, \infty)$ and define

$$M = \sup_{t \geqslant 0} |x(t)|.$$

Let $\epsilon > 0$ be given; we may suppose ϵ so small that $u(\epsilon) < v(M)$. Then there exists a number $a = a(\epsilon) > 0$, such that $f(s) - s > a$ for $s \in [u(\epsilon), v(M)]$. Let $N = N(\epsilon) > 0$ be the smallest integer such that $v(M) \leqslant u(\epsilon) + Na$, and define $\epsilon_j = u(\epsilon) + (N - j)a$, $j = 0, 1, 2,..., N$. We note that $V(t, x(t)) \leqslant \epsilon_0$ for $t \geqslant 0$.

Suppose $V(t, x(t)) \geqslant \epsilon_1$ for all $t \geqslant r$. Then for any such t, $v(|x(t)|) \geqslant \epsilon_1$, and hence $|x(t)| \geqslant v^{-1}(\epsilon_1) > 0$. Also for such t, $u(\epsilon) \leqslant V(t, x(t)) \leqslant v(M)$, so that

$$f(V(t, x(t))) > V(t, x(t)) + a \geqslant u(\epsilon) + (N - 1)a + a = u(\epsilon) + Na.$$

But $V(s, x(s)) \leqslant u(\epsilon) + Na$ for all $s \geqslant 0$, and thus also for $s \in [t - r, t]$, $t \geqslant r$. Using (b) with $j = 0$, we conclude that

$$\dot{V}(t, x(t)) \leqslant -w(|x(t)|), \qquad t \geqslant r. \tag{3}$$

Define $\rho_1 = v^{-1}(\epsilon_1)$, and $\gamma_1 = \inf_{s \in [\rho_1, M]} w(s) > 0$; it follows then from (3) that

$$V(t, x(t)) \leqslant V(r, x(r)) - \gamma_1(t - r) \leqslant \epsilon_0 - \gamma_1(t - r),$$

for all $t \geqslant r$. Since $V(t, x(t))$ is never negative, this leads to a contradiction.

So there exists a $t_1 > r$ such that $V(t_1, x(t_1)) < \epsilon_1$. If $V(\tilde{t}_1, x(\tilde{t}_1)) = \epsilon_1$ for some $\tilde{t}_1 > t_1$, we may suppose $\tilde{t}_1$ chosen so that $V(t, x(t)) < \epsilon_1$ for $t \in [t_1, \tilde{t}_1)$, and it follows clearly that

$$\dot{V}(\tilde{t}_1, x(\tilde{t}_1)) \geqslant 0. \tag{4}$$

However,

$$f(\epsilon_1) = f(V(\tilde{t}_1, x(\tilde{t}_1))) > V(\tilde{t}_1, x(\tilde{t}_1)) + a = \epsilon_1 + a = \epsilon_0 .$$

Since also $V(s, x(s)) \leqslant \epsilon_0$ for $s \in [\tilde{t}_1 - r, \tilde{t}_1]$, it follows from (b) that

$$\dot{V}(\tilde{t}_1, x(\tilde{t}_1)) \leqslant -w(|\, x(\tilde{t}_1)|) < 0.$$

This contradicts (4), so we must conclude that

$$V(t, x(t)) < \epsilon_1 \qquad \text{for all} \quad t \geqslant t_1 . \tag{5}$$

Suppose $V(t, x(t)) \geqslant \epsilon_2$ for all $t \geqslant t_1$. Then for $t \geqslant t_1 + r$, we have $v(|\, x(t)|) \geqslant \epsilon_2$, and hence $|\, x(t)| \geqslant v^{-1}(\epsilon_2)$; define $\rho_2 = v^{-1}(\epsilon_2)$. Since $\epsilon_2 \leqslant V(t, x(t)) \leqslant \epsilon_1$ for $t \geqslant t_1 + r$, it follows that for such t,

$$f(V(t, x(t))) > V(t, x(t)) + a \geqslant u(\epsilon) + (N - 1)a = \epsilon_1 \geqslant V(s, x(s))$$

for $s \in [t - r, t]$. So by (b) we have

$$\dot{V}(t, x(t)) \leqslant -w(|\, x(t)|), \qquad t \geqslant t_1 + r. \tag{6}$$

If $\gamma_2 = \inf_{s \in [\rho_2, M]} w(s)$, then $\gamma_2 > 0$, and from (6) we have

$$V(t, x(t)) \leqslant V(t_1 + r, x(t_1 + r)) - \gamma_2(t - t_1 - r) \leqslant \epsilon_1 + \gamma_2(t - t_1 - r).$$

But for $t \geqslant t_1 + r$ and sufficiently large, this leads to a contradiction. So there exists $t_2 \geqslant t_1 + r$ such that $V(t_2, x(t_2)) < \epsilon_2$.

Suppose for some $\tilde{t}_2 > t_2$, $V(\tilde{t}_2, x(\tilde{t}_2)) = \epsilon_2$, while $V(t, x(t)) < \epsilon_2$ for $t \in [t_2, \tilde{t}_2)$. It follows that

$$\dot{V}(\tilde{t}_2, x(\tilde{t}_2)) \geqslant 0. \tag{7}$$

However,

$$f(\epsilon_2) = f(V(\tilde{t}_2, x(\tilde{t}_2))) > V(\tilde{t}_2, x(\tilde{t}_2)) + a = \epsilon_2 + a = \epsilon_1 .$$

But also $V(s, x(s)) \leqslant \epsilon_1$ for $s \in [\tilde{t}_2 - r, \tilde{t}_2]$: this follows from (5) since for such s, $s \geqslant t_2 - r \geqslant t_1$. So $f(V(\tilde{t}_2, x(\tilde{t}_2))) > V(s, x(s))$ for $s \in [\tilde{t}_2 - r, \tilde{t}_2]$, and using (b), we obtain $\dot{V}(\tilde{t}_2, x(\tilde{t}_2)) < 0$, which contradicts (7). So we must have $V(t, x(t)) < \epsilon_2$ for $t \geqslant t_2$.

Continuing in this way, we get for $j = 0, 1, \ldots, N$, that there exists t_j such that $V(t, x(t)) < \epsilon_j$ for $t \geqslant t_j$, where $t_j \geqslant t_{j-1} + r$, and $t_0 = 0$. But $\epsilon_N = u(\epsilon)$: i.e., $V(t, x(t)) < u(\epsilon)$ for $t \geqslant t_N$. Thus for such t, we have

$$u(|x(t)|) < u(\epsilon),$$

from which we get $|x(t)| < \epsilon$ for $t \geqslant t_N$. This proves the theorem.

In systems such as (1.1) and more generally, (1), the solution is a function on $[0, T)$ to R^n and the equation specifies the derivative of this solution for all $t \in [0, T)$. We can easily generalize our theorem to the case where the equation still specifies the derivative of the solution on $[0, T)$, but where the solution must be defined on $[-r, T)$ for some fixed $r > 0$, and where its derivative at t depends on the solution $x(s)$, $s \in [-r, t]$, $t \in [0, T)$. For example, we can consider a system of type

$$\dot{x}(t) = f(t, x_t) + \int_0^t g(t, s, x(s))\, ds, \tag{1.2}$$

where g is as in (1.1), $f(t, \phi)$ is a continuous function on $R \times C_r$, C_r the space (with the usual supremum norm) of continuous functions on $[-r, 0]$ to R^n, and if $x(t)$ is a solution of (1.2) it must be continuous on $[-r, T)$, continuously differentiable on $(0, T)$, and satisfy (1.2) there where x_t denotes the element of C_r defined by $x(t + s)$, $s \in [t - r, t]$. The part of the solution for $t \in [-r, 0]$ can be regarded as the initial value for the solution. Thus we now include systems which are referred to as functional differential equations of retarded type, cf. [4], to which Razumikhin originally applied his method.

The following theorem applies to (1.2), and involves much stronger conditions which are however more easily applicable to more explicit cases of (1.2), as our final example will show.

THEOREM 2. *Suppose that*

(a_1) *there exists a function V on $R \times R^n$ to R satisfying* (*a*) *and having continuous first partial derivatives in all variables there*;

(b_1) *there exist functions f and w satisfying the same conditions as in* (b) *except that w is also increasing*;

(c_1) *given $\epsilon > 0$ and $M > 0$, there exists a $k = k(\epsilon, M) > 0$ such that*

$$\text{(i)} \quad \frac{\partial V}{\partial x} \cdot \int_0^{t-kr} g(t, s, x)\, ds \leqslant \epsilon \qquad \textit{for} \quad t \geqslant kr, \quad |x| \leqslant M;$$

and if $x(t)$ is a solution of (1.2) *satisfying*

$$\text{(ii)} \quad V(s, x(s)) < f(V(t, x(t))) \qquad \textit{for} \quad t - kr \leqslant s \leqslant t, \quad t \geqslant kr,$$

then

$$\frac{\partial V}{\partial x} \cdot \left[f(t, x_t) + \int_{t-kr}^{t} g(t, s, x(s))\, ds\right] + \frac{\partial V}{\partial t} \leqslant -w(|\, x(t)|)$$

for $t > kr$.

Then if $x(t)$ *is a bounded solution of* (1.2), $x(t) \to 0$ *as* $t \to +\infty$.

$$\text{In } (c_1), \qquad \frac{\partial V}{\partial x} = \left(\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}\right).$$

Proof of Theorem 2. Let $\epsilon > 0$ be given, and $x(t)$ be a bounded solution of (1.2). Let $N = N(\epsilon)$ and ϵ_j, $j = 0, 1, \ldots, N$, be as defined in the proof of Theorem 1. Then using (c_1), (i), there exists for each integer j, $0 \leqslant j \leqslant N$, a $k_j = k_j(\epsilon_j, M)$ such that

$$\frac{\partial V}{\partial x} \cdot \int_0^{t-k_j r} g(t, s, x)\, ds \leqslant w(v^{-1}(\epsilon_j))/2 \tag{8}$$

for $t \geqslant k_j r$ and $|\, x\,| \leqslant M = \sup_{t \geqslant 0} |\, x(t)|$: here v^{-1} is the inverse of v defined in (a).

As in the proof of Theorem 1, we wish to show that $V(t, x(t)) < u(\epsilon) = \epsilon_N$ for all t sufficiently large. Since $V(t, x(t)) < \epsilon_0$ for all $t \geqslant 0$, and $\epsilon_j > \epsilon_{j+1}$, $j = 0, \ldots, N-1$, it follows that there exists a greatest integer j_0, $0 \leqslant j_0 \leqslant N$, such that $V(t, x(t)) < \epsilon_{j_0}$ for all t sufficiently large. Suppose $j_0 < N$ and suppose that $V(t, x(t)) \geqslant \epsilon_{j_0+1}$ for all $t \geqslant r_0 \equiv k_{j_0+1} r$ sufficiently large. Then as in the proof of Theorem 1, we find that

$$V(s, x(s)) < f(V(t, x(t))), \qquad t - r_0 \leqslant s \leqslant t,$$

and all $t \geqslant r_0$ sufficiently large. Using (c_1), we then have

$$\begin{aligned} \dot{V}(t, x(t)) &= \frac{\partial V}{\partial x} \cdot \left[f(t, x_t) + \int_{t-r_0}^{t} g(t, s, x(s))\, ds \right. \\ &\qquad \left. + \int_0^{t-r_0} g(t, s, x(s))\, ds\right] + \frac{\partial V}{\partial t} \\ &\leqslant -w(|\, x(t)|) + w(v^{-1}(\epsilon_j))/2 \\ &= -w(|\, x(t)|)/2 \end{aligned} \tag{8.1}$$

for $t \geqslant r_0$ and sufficiently large. Using this and the fact that

$$|\, x(t)| \geqslant v^{-1}(\epsilon_{j_0+1}) > 0$$

for such t, we obtain, as in the proof of Theorem 1, a contradiction.

So there exists a $t_0 \geqslant r_0$ such that $V(t_0, x(t_0)) < \epsilon_{j_0+1}$. If $V(t, x(t)) < \epsilon_{j_0+1}$ for $t \geqslant t_0$, we contradict our choice of j_0. So there exists a $\tilde{t}_0 > t_0$ such that $V(\tilde{t}_0, x(\tilde{t}_0)) = \epsilon_{j_0+1}$ and $\dot{V}(\tilde{t}_0, x(\tilde{t}_0)) \geqslant 0$. But as in the proof of Theorem 1, we find that $f(V(\tilde{t}_0, x(\tilde{t}_0))) > V(s, x(s))$ for $\tilde{t}_0 - r_0 \leqslant s \leqslant \tilde{t}_0$. As above, we use ($c_1$) again and find, as (8.1), that $\dot{V}(\tilde{t}_0, x(\tilde{t}_0)) < 0$, again a contradiction. We can only conclude that $j_0 = N$, which essentially completes the proof of Theorem 2.

As an example of this last result, we consider the system

$$\dot{x}(t) = Ax(t) + h(t, x_t) + \int_0^t g(t, s, x(s))\, ds, \tag{1.3}$$

where A is a real stable $n \times n$ matrix, h is a function continuous on $R \times C_r$ and satisfying $|h(t, \phi)| \leqslant \mu \|\phi\|$, where

$$\|\phi\| = \sup_{s \in [-r, 0]} |\phi(s)|,$$

and $g(t, s, x)$ is continuous on $R \times R \times R^n$ for $t \geqslant s \geqslant 0$, and satisfies $|g(t, s, x)| \leqslant K(t, s)|x|$ with

$$\int_0^t K(t, s)\, ds \to 0 \qquad \text{as} \quad t \to +\infty.$$

We show that for μ sufficiently small, each bounded solution of (1.3) tends to zero as $t \to +\infty$.

Since A is a stable matrix, by a well-known result, there exists a positive definite symmetrix matrix B such that $BA + A^T B = -I$, here A^T denotes the transpose of A and I is the identity matrix. Define $V = Bx \cdot x$.

To show that (b_1) holds for μ sufficiently small, we observe first that there exist positive numbers λ and Λ such that $\lambda^2 |x|^2 \leqslant Bx \cdot x \leqslant \Lambda^2 |x|^2$ for all $x \in R^n$. If

$$\mu < \lambda/(2 |B| \Lambda), \tag{9}$$

where $|B| = \sum_{i,j=1}^n |b_{ij}|$, $B = (b_{ij})$, then there exists a $q > 1$, such that

$$\mu < \lambda/(2q |B| \Lambda).$$

We choose $f(s) = q^2 s$. Then for any positive integer k and any solution $x(t)$ of (1.3) such that

$$Bx(s) \cdot x(s) < q^2 Bx(t) \cdot x(t) \qquad \text{for} \quad s \in [t - kr, t], \quad t \geqslant kr, \tag{10}$$

we have clearly $q^2 \Lambda^2 |x(t)|^2 > \lambda^2 \|x_t\|^2$, $t \geqslant kr$. Thus

$$\begin{aligned} |Bx(t) \cdot h(t, x_t)| &\leqslant |B|\, |x(t)|\, \mu \|x_t\| \\ &\leqslant \mu |B| q\Lambda |x(t)|^2/\lambda, \quad t \geqslant kr. \end{aligned} \tag{11}$$

Fix $\mu_1 > 0$ such that

$$(2 \mid B \mid qA/\lambda)(\mu + \mu_1) < 1, \tag{12}$$

then there exists a $k > 0$ such that for $t \geqslant kr$,

$$\int_0^t K(t, s)\, ds \leqslant \mu_1 .$$

Since

$$\left| \int_{t-kr}^{t} g(t, s, x(s))\, ds \right| \leqslant \int_{t-kr}^{t} K(t, s) \mid x(s) \mid ds$$
$$\leqslant \sup_{\theta \in [t-kr, t]} \mid x(\theta) \mid \int_0^t K(t, s)\, ds,$$

$t \geqslant kr$, and $x(t)$ a solution of (1.3) satisfying (10), it follows that for such k, t, and $x(t)$,

$$\left| 2Bx(t) \cdot \int_{t-kr}^{t} g(t, s, x(s))\, ds \right| \leqslant 2qA\mu_1 \mid B \mid \mid x(t) \mid^2/\lambda. \tag{13}$$

If we define $\alpha = 1 - 2 \mid B \mid qA(\mu + \mu_1)/\lambda$, and $w(s) = \alpha s^2$, we conclude that (ii) of c_1) holds, note that (12) implies $\alpha > 0$.

It remains to show that (i) of (c_1) also holds; but this follows easily from the fact that

$$\left| \int_0^{t-kr} g(t, s, x)\, ds \right| \leqslant \mid x \mid \int_0^t K(t, s)\, ds \to 0 \qquad \text{as} \quad t \to +\infty.$$

Hence under the hypotheses given above on A, h, and g, if (9) holds, then each bounded solution of (1.3) tends to zero as $t \to +\infty$.

A somewhat less restrictive condition on g which yields the same conclusion is to suppose that $\mid g(t, s, x) \mid \leqslant K(t, s) \mid x \mid$ with

$$\int_0^t K(t, s)\, ds \leqslant \mu_1 \qquad \text{for} \quad t \geqslant 0,$$

where μ_1 satisfies (12), and that

$$\int_0^t K(t + \tau, s)\, ds \to 0 \qquad \text{as} \quad \tau \to +\infty, \tag{14}$$

the limit being uniform for $t \geqslant 0$. It is clear that (14) implies condition (c_1); we omit the details.

APPENDIX

Consider the scalar equation

$$\dot{x}(t) = -2x(t) + x(0). \tag{1.3}$$

It is clearly of type (1), and all its solutions are of the form

$$x(t) = (1 + e^{-2t})\, x(0)/2.$$

Clearly $x = 0$ is not asymptotically stable for this equation. However, if we take $V(t, x) = x^2/2$ and $f(s) = 2s$, we find that

$$\dot{V}_{(1.3)}(x(t)) = -2x^2(t) + x(t)\, x(0). \tag{1.4}$$

If $x(t)$ is a solution such that $f(V(x(t))) > V(x(s))$ for $0 \leqslant s \leqslant t$, $t > 0$, then clearly $\sqrt{2}\, |\, x(t)| > |\, x(0)|$. It follows from (1.4) that for such solutions $x(t)$

$$\dot{V}_{(1.3)}(x(t)) \leqslant (-2 + \sqrt{2})\, x^2(t),$$

i.e., $\dot{V}_{(1.3)}$ is bounded above by a negative definite function of $|\, x(t)|$. It follows that while such a condition is sufficient for the asymptotic stability of $x = 0$ for systems with a fixed finite delay interval (cf., for example, Theorem 11.2 in [4]), it is clearly insufficient for systems like (1).

This example also motivates condition (b) in the hypotheses of Theorem 1. Loosely speaking, this condition says that if a solution $x(t)$ of (1) ever is such that $V(t, x(t))$ is too close to the supremum of $V(s, x(s))$ as s ranges from $t - r$ to t, and $t > r$, then $V(t, x(t))$ must decrease as t increases. However, if $V(t, x(t))$ is too close to the supremum of $V(s, x(s))$ as s ranges from 0 to t, then our example shows that a condition such as

$$\dot{V}(t, x(t)) \leqslant -\alpha(|\, x(t)|),$$

where α is continuous and positive definite, is not sufficient for asymptotic stability of $x = 0$, even though such a condition implies that $V(t, x(t))$ must decrease as t increases.

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