STP Toolbox for Matlab*

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1 Introduction

The STP toolbox is developed for calculating the semi-tensor product (STP) of (logical) matrices and its application to the analysis and control of Boolean networks.

The semi-tensor product of matrices is defined as follows [1, 2]

Definition 1.1 *1. Let X be a row vector of dimension np, and Y be a column vector with dimension p. Then we split X into p equal-size blocks as $X^1, \cdots, X^p$, which are $1 \times n$ rows. Define the STP, denoted by $\ltimes$, as*

$$\begin{cases} X \ltimes Y = \sum_{i=1}^{p} X^i y_i \in \mathbb{R}^n, \\ Y^T \ltimes X^T = \sum_{i=1}^{p} y_i (X^i)^T \in \mathbb{R}^n. \end{cases} \tag{1}$$

2. Let $A \in M_{m\times n}$ and $B \in M_{p\times q}$. If either n is a factor of p, say $nt = p$ and denote it as $A \prec_t B$, or p is a factor of n, say $n = pt$ and denote it as $A \succ_t B$, then we define the STP of A and B , denoted by $C = A \ltimes B$, as the following: C consists of $m \times q$ blocks as $C = (C^{ij})$ and each block is

$$C^{ij} = A^i \ltimes B_j, \quad i = 1, \cdots, m, \quad j = 1, \cdots, q,$$

where A^i is i-th row of A and B_j is the j-th column of B.

We use some simple numerical examples to describe it.

Example 1.2 1. Let $X = \begin{bmatrix} 1 & 2 & 3 & -1 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then

$$X \ltimes Y = \begin{bmatrix} 1 & 2 \end{bmatrix} \cdot 1 + \begin{bmatrix} 3 & -1 \end{bmatrix} \cdot 2 = \begin{bmatrix} 7 & 0 \end{bmatrix}.$$

2. Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 1 & 2 \\ 3 & 2 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}.$$

Then

$$A \ltimes B = \begin{bmatrix} \begin{pmatrix} 1 & 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ -1 \end{pmatrix} \\ \begin{pmatrix} 2 & 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} & \begin{pmatrix} 2 & 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} -2 \\ -1 \end{pmatrix} \\ \begin{pmatrix} 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} & \begin{pmatrix} 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ -1 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} 3 & 4 & -3 & -5 \\ 4 & 7 & -5 & -8 \\ 5 & 2 & -7 & -4 \end{bmatrix}.$$

□

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Definition 1.3 *1. An $n \times p$ matrix, A, is called a logical matrix if*

$$A = \left[\delta_n^{i_1} \ \delta_n^{i_2} \ \cdots \ \delta_n^{i_p}\right], \tag{2}$$

where δ_n^i is the i-th column of the identity matrix I_n.

2. The condense form of a logical matrix (as A in (2)) is denoted as

$$A = \delta_n[i_1, i_2, \cdots, i_p]. \tag{3}$$

Remark 1.4 *According to (3), an $n \times p$ logical matrix is described by a vector of dimension p and a parameter n. In the toolbox structure "`lm`" is used to express a logical matrix as*

$$\begin{cases} lm.n = n, \\ lm.v = [i_1, i_2, \cdots, i_p]. \end{cases} \tag{4}$$

Example 1.5 *Consider*

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is a logical matrix and it can be expressed in condensed form as

$$A = \delta_4[1, 3, 2, 4].$$

Expressing A in "`lm`" structure, we have

$$\begin{cases} A.n = 4, \\ A.v = [1 \ 3 \ 2 \ 4]. \end{cases}$$

□

Definition 1.6 *The swap matrix $W_{[m,n]}$ is an $mn \times mn$ matrix constructed in the following way: label its columns by $(11, 12, \cdots, 1n, \cdots, m1, m2, \cdots, mn)$ and its rows by $(11, 21, \cdots, m1, \cdots, 1n, 2n, \cdots, mn)$. Then its element in the position $((I, J), (i, j))$ is assigned as*

$$w_{(IJ),(ij)} = \delta_{i,j}^{I,J} = \begin{cases} 1, & I = i \text{ and } J = j, \\ 0, & \text{otherwise}. \end{cases} \tag{5}$$

Remark 1.7 *Let $X \in \mathbb{R}^m$ and $Y \in \mathbb{R}^n$ be two columns. Then*

$$W_{[m,n]} \ltimes X \ltimes Y = Y \ltimes X. \tag{6}$$

Example 1.8 *Let $m = 2$ and $n = 3$, the swap matrix $W_{[2,3]}$ is constructed as*

$$W_{[2,3]} = \begin{array}{c} \begin{array}{cccccc} (11) & (12) & (13) & (21) & (22) & (23) \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ (11) \\ (21) \\ (12) \\ (22) \\ (13) \\ (23) \end{array}.$$

In condensed form we have

$$W_{[2,3]} = \delta_6[1, 3, 5, 2, 4, 6].$$

□

Definition 1.9 *Let A be an $m \times n$ matrix, $m = pq$, $n = rs$. Express A in blocks as*

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & & & \\ A_{q1} & A_{q2} & \cdots & A_{qs} \end{bmatrix}, \tag{7}$$

where $A_{i,j}$ are $p \times r$ matrices. then the block transpose $A^{T(p,r)}$ is defined as

$$A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{q1} \\ A_{12} & A_{22} & \cdots & A_{q2} \\ \vdots & & & \\ A_{1s} & A_{2s} & \cdots & A_{qs} \end{bmatrix}, \tag{8}$$

2 Functions

This section provides detailed description for the functions in this package.

2.1 Fundamental Calculations

1. $C = sp(A, B)$

   Description: The function performs the semi-tensor product of two matrices A and B.

   Argument(s): Two matrices A and B, and the column number of A must be the factor or multiple of the row number of B.

   Return Value: $C = A \ltimes B$.

2. $C = sp1(A, B)$

   Description: The function performs the semi-tensor product of two matrices A and B.

   Argument(s): Two matrices A and B, and the column number of a must be the factor or multiple of the row number of b.

   Return Value: $C = A \ltimes B$.

   Note that this function is the same as the function $C = sp(A, B)$. The difference is inside. In program this function uses the original definition, while $C = sp(A, B)$ uses the Kronecker product according to some properties.

3. $C = spn(A_1, A_2, \cdots, A_n)$

   Description: The function performs the semi-tensor product of finite set of matrices $A_1, \cdots, A_n$.

   Argument(s): Finite matrices $A_1, \cdots, A_n$ which are of proper dimension.

   Return Value: $C = \ltimes_{i=1}^{n} A_i$.

4. $B = bt(A, p, r)$

   Description: The function performs the block transpose of A (refer to Definition 1.9 for the definition).

   Argument(s): A is the matrix to be transposed; the size of fixed blocks is $p \times r$ (refer to Definition 1.9 for the concept.)

   Return Value: $B = A^{T(p,r)}$.

5. $W = wij(m, n)$

   Description: This function produces an $mn \times mn$ swap matrix (refer to Definition 1.6 for the definition).

   Argument(s): Two positive integers m and n. n is optional, default n is m.

   Return Value: Matrix W of dimension $mn \times mn$.

6. $v = vc(A)$

   Description: The function converts a matrix to its column stacking form.

   Argument(s): Matrix $A = (a_{ij})_{m \times n}$.

   Return Value: $v = [a_{11} \ \cdots \ a_{m1} \ \cdots \ a_{1n} \ \cdots \ a_{mn}]^T$.

7. $v = vr(A)$

   Description: The function converts a matrix to its row stacking form.

   Argument(s): Matrix $A = (a_{ij})_{m \times n}$.

   Return Value: $v = [a_{11} \ \cdots \ a_{1n} \ \cdots \ a_{m1} \ \cdots \ a_{mn}]^T$.

8. $A = invvc(x, m)$

   Description: Let $x = (x_1, x_2, \cdots, x_p)$. This function reshapes x into a matrix A with row number m as

$$A = \begin{bmatrix} x_1 & x_{m+1} & \cdots & x_{p-m+1} \\ x_2 & x_{m+2} & \cdots & x_{p-m+2} \\ \vdots & & & \\ x_m & x_{2m} & \cdots & x_p \end{bmatrix}.$$

   If p is not a multiple of m, we add at the end of x a least number of zeros such that the length of x becomes a multiple of m.

   Argument(s): x is a vector; m is the row number of the resulting matrix, and it is optional. Default m is $ceil(sqrt(length(v)))$.

   Return Value: Matrix A with row number m.

9. $A = invvr(x, n)$

   Description: Let $x = (x_1, x_2, \cdots, x_p)$. This function reshapes x into a matrix A with column number n as

$$A = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ x_{n+1} & x_{n+2} & \cdots & x_{2n} \\ \vdots & & & \\ x_{p-n+1} & x_{p-n+2} & \cdots & x_p \end{bmatrix}.$$

   If p is not a multiple of n, we add at the end of x a least number of zeros such that the length of x becomes a multiple of n.

   Argument(s): x is a vector; m is the column number of the resulting matrix, and it is optional. Default m is $ceil(sqrt(length(v)))$.

   Return Value: Matrix A with column number m.

10. $v = dec2any(a, k, len)$

    Description: The function converts a decimal number a into a k-based number as

$$a = a_s k^s + a_{s-1} k^{s-1} + \cdots + a_1 k + a_0, \quad a_s > 0.$$

(Say, $k = 2$, the result is a binary number. In fact we define $v = dec2bin(a, len)$ for binary case.)

Argument(s): a is a positive integer; k is optional, and $k \geq 2$. Default k is 2. Default len is 0, it means $a_s \neq 0$, but if $len > 0$ and $len > s+1$, $len - s - 1$ zeros should be added at the beginning of returned value.

Return Value: $v = [a_s \; a_{s-1} \; \cdots \; a_1 \; a_0]$.

11. $M = stp(A)$

    Description: `stp` class constructor.

    Argument(s): Matrix A.

    Return Value: `stp` object M.

2.2 Calculation for Logical Matrices

In this subsection we will introduce the functions for logical matrices or `lm` structure.

1. $M = lm(A)$ or $M = lm(v, n)$

   Description: `lm` class constructor.

   Argument(s): i) Logical matrix A; ii) vector $v = [v_1 \; v_2 \; \cdots \; v_p]$ and positive integer n satisfying $v_i \leq n, \; 1 \leq i \leq p$. (Case i, refer to Definiation 1.3 and Example 1.5; Case ii, $lm.n = n, \; lm.v = v$.)

   Return Value: `lm` object M.

2. $C = lsp(A, B)$

   Description: The function performs the semi-tensor product of logcial matrices A and B.

   Argument(s): A, B are `lm` objects. (refer to Definition 1.3 and Remark 1.4 for the structure.)

   Return Value: $C = A \ltimes B$ is an `lm` object.

3. $C = lspn(A_1, A_2, \cdots, A_n)$

   Description: The function performs the semi-tensor product of logical matrices $A_1, A_2, \cdots, A_n$.

   Argument(s): $A_1, \cdots, A_n$ are `lm` objects. (refer to Definition 1.3 and Remark 1.4 for the structure.)

   Return Value: $C = \ltimes_{i=1}^{n} A_i$ is an `lm` object.

4. $M = leye(n)$

   Description: The function produces an $n \times n$ identity matrix.

   Argument(s): Positive integer n.

   Return Value: `lm` object M.

5. $M = lmn(k)$

   Description: The function produces the structure matrix of negation for k-valued logic ($k \geq 2$).

   Argument(s): k is optional, default k is 2.

   Return Value: `lm` object M.

6. $M = lmc(k)$

   Description: The function produces the structure matrix of conjunction for k-valued logic ($k \geq 2$).

   Argument(s): k is optional, default k is 2.

   Return Value: `lm` object M.

7. $M = lmd(k)$

   Description: The function produces the structure matrix of disjunction for k-valued logic ($k \geq 2$).

   Argument(s): k is optional, default k is 2.

   Return Value: `lm` object M.

8. $M = lmi(k)$

   Description: The function produces the structure matrix of implication for k-valued logic ($k \geq 2$).

   Argument(s): k is optional, default k is 2.

   Return Value: `lm` object M.

9. $M = lme(k)$

   Description: The function produces the structure matrix of equivalance for k-valued logic ($k \geq 2$).

   Argument(s): k is optional, default k is 2.

   Return Value: `lm` object M.

10. $M = lmr(k)$

    Description: The function produces the power-reducing matrix for k-valued logic ($k \geq 2$).

    Argument(s): k is optional, default k is 2.

    Return Value: `lm` object M.

11. $M = lmu(k)$

    Description: The function produces the dummy matrix for k-valued logic ($k \geq 2$). The dummy matrix M satisfies the following property

    $$MXY = Y, \quad \forall X, Y \in D_k.$$

    Argument(s): k is optional, default k is 2.

    Return Value: `lm` object M.

12. $M = lmrand(m, n)$

    Description: The function produces an $m \times n$ logical matrix randomly.

    Argument(s): Positive integers m and n. n is optional, default n is m.

    Return Value: `lm` object M.

13. $M = lwij(m, n)$

    Description: The function produces an $mn \times nn$ swap matrix.

    Argument(s): Positive integers m and n. n is optional, default n is m.

    Return Value: `lm` object M.

14. $M = randlm(m, n)$

    Description: Alias function of *lmrand*.

3 Examples

```
% This example is to show how to perform semi-tensor product

x = [1 2 3 -1];
y = [2 1]';
r1 = sp(x,y)
% r1 = [5, 3]

x = [2 1];
y = [1 2 3 -1]';
r2 = sp(x,y)
% r2 = [5; 3]

x = [1 2 1 1;
     2 3 1 2;
     3 2 1 0];
y = [1 -2;
     2 -1];
r3 = sp(x,y)
r4 = sp1(x,y)
% r3 = r4 = [3,4,-3,-5;4,7,-5,-8;5,2,-7,-4]

r5 = sp(sp(x,y),y)
r6 = spn(x,y,y)
% r5 = r6 = [-3,-6,-3,-3;-6,-9,-3,-6;-9,-6,-3,0]
```

```
% This example is to show the usage of stp class.
% Many useful methods are overloaded for stp class, thus you can use stp object as double.

x = [1 2 1 1;
     2 3 1 2;
     3 2 1 0];
y = [1 -2;
     2 -1];

% Covert x and y to stp class
a = stp(x);
b = stp(y);

% mtimes method is overloaded by semi-tensor product for stp class
c0 = spn(x,y,y)
c = a*b*b, class(c)

% Convert an stp object to double
c1 = double(c), class(c1)

% size method for stp class
size(c)

% length method for stp class
length(c)

% subsref method for stp class
c(1,:)

% subsasgn method for stp class
```

```
c(1,1) = 3
```

```
% This example is to show the usage of lm class.
% Many methods are overloaded for lm class.

% Consider classical (2-valued) logic here
k = 2;

T = lm(1,k); % True
F = lm(k,k); % False

% Given a logical matrix, and convert it to lm class
A = [1 0 0 0;
     0 1 1 1]
M = lm(A)
% or we can use
% M = lm([1 2 2 2], 2)

% Use m-function to perform semi-tensor product for logical matrices
r1 = lspn(M,T,F)

% Use overloaded mtimes method for lm class to perform semi-tensor product
r2 = M*T*F

% Create an 4-by-4 logical matrix randomly
M1 = lmrand(4)
% M1 = randlm(4)

% Convert an lm object to double
double(M1)

% size method for lm class
size(M1)

% diag method for lm class
diag(M1)

% Identity matrix is a special type of logical matrix
I3 = leye(3)

% plus method is overloaded by Kronecher product for lm class
r3 = M1 + I3
% Alternative way to perform Kronecher product of two logical matrices
r4 = kron(M1,I3)

% Create an lm object by assignment
M2 = lm;
M2.n = 2;
M2.v = [1 1 2 2];
M2
```

```
% This example is to show how to use vector form of logic to solve the following question:
% A said B is a liar, B said C is a liar, and C said A and B are both liars. Who is the liar?

% Set A: A is honest, B: B is honest, C: C is honest

```

```
k = 2;        % Two-valued logic
MC = lmc(k); % structure matrix for conjunction
ME = lme(k); % structure matrix for equivalance
MN = lmn(k); % structure matrix for negation
MR = lmr(k); % power-reducing matrix

% The logical expression can be written as
logic_expr = '(A=!B)&(B=!C)&(C=(!A&!B))';
% where = is equivalance, & is conjunction, and ! is negation

% convert the logic expresson to its matrix form
matrix_expr = lmparser(logic_expr);

% then obtain its canonical matrix form
expr = stdform(matrix_expr);

% calculate the structure matrix
L = eval(expr)

% The uniqe solution for L*x=[1 0]^T is x=[0 0 0 0 0 1 0 0]^T:=8[6]
sol = v2s(lm(6,8))

% One can see sol=[0 1 0], which means only B is honest, A and C are liars.
```

```
% Examples for Boolean network

% Initializing
k = 2;
options = [];

% Please note that in this toolbox any variable intialized with capital M is defined
    as a logical matrix, otherwise it will be considered as logical vector.
% The followings are some commonly used logical matrices
ME = lme(k); % equivalence
MI = lmi(k); % implicaiton
MD = lmd(k); % disjunction
MN = lmn(k); % negation
MR = lmr(k); % power-reducing matrix
MC = lmc(k); % conjunction
MX = lm([2 1 1 2], 2); % xor

% choose a number from 1-5 to select a Boolean network
n = 3;

switch n
    case 1
        % Dynamics of Boolean network
            % A(t+1) = MC*B(t)*C(t)
            % B(t+1) = MN*A(t)
            % C(t+1) = MD*B(t)*C(t)
        % Set X(t)=A(t)B(t)C(t), then
        eqn = 'MC B C MN A MD B C';
    case 2
        % Dynamics of Boolean network
        % A(t+1) = MC*B(t)*C(t)
        % B(t+1) = MN*A(t)
        % C(t+1) = B(t)
        eqn = 'MC B C MN A B';
    case 3
        % Dynamics of Boolean network
```

```
            % E(t+1) = MX*E(t)*I(t)
            % H(t+1) = MX*F(t)*H(t)
            % F(t+1) = MX*F(t)*J(t)
            % I(t+1) = MX*G(t)*I(t)
            % G(t+1) = MX*G(t)*MX*F(t)*H(t)
            % J(t+1) = MX*MX*E(t)*I(t)*J(t)
        % Set X(t)=E(t)H(t)F(t)I(t)G(t)J(t), then
        if k ≠ 2
            error('This example is only for the case k=2.');
        end
        eqn = 'MX E I MX F H MX F J MX G I MX G MX F H MX MX E I J ';
        % set the variables' order, otherwise they will be sorted in the dictionary order
        options = lmset('vars',{'E','H','F','I','G','J'});
    case 4
        % Dynamics of Boolean network
            % A(t+1) = MN*MI*K(t)*H(t)
            % B(t+1) = MN*MI*A(t)*C(t)
            % C(t+1) = MI*D(t)*I(t)
            % D(t+1) = MC*J(t)*K(t)
            % E(t+1) = MI*C(t)*F(t)
            % F(t+1) = MN*MI*E(t)*G(t)
            % G(t+1) = MN*MC*B(t)*E(t)
            % H(t+1) = MN*MI*F(t)*G(t)
            % I(t+1) = MN*MI*H(t)*I(t)
            % J(t+1) = J(t)
            % K(t+1) = K(t)
        % Set X(t)=A(t)B(t)C(t)D(t)E(t)F(t)G(t)H(t)I(t)J(t)K(t), then
        eqn = 'MN MI K H MN MI A C MI D I MC J K MI C F MN MI E G MN MC B E MN MI F G MN MI H I J K';
    case 5
        % Dynamics of Boolean network
            % A(t+1) = MN*MD*C(t)*F(t)
            % B(t+1) = A(t)
            % C(t+1) = B(t)
            % D(t+1) = MC*MC*MN*I(t)*MN*C(t)*MN*F(t)
            % E(t+1) = D(t)
            % F(t+1) = E(t)
            % G(t+1) = MN*MD*F(t)*I(t)
            % H(t+1) = G(t)
            % I(t+1) = H(t)
        % Set X(t)=A(t)B(t)C(t)D(t)E(t)F(t)G(t)H(t)I(t), then
        eqn = 'MN MD C F A B MC MC MN I MN C MN F D E MN MD F I G H';
    otherwise
        return
end

% Convert the equation to a canonical form
[expr,vars] = stdform(eqn,options,k);

% Calculate the network transition matrix
L = eval(expr)

% Analyze the dynamics of the Boolean network
[n,l,c,r0,T] = bn(L,k);

fprintf('Number of attractors: %d\n\n',n);
fprintf('Lengths of attractors:\n');
disp(l);
fprintf('\nAll attractors are displayed as follows:\n\n');
for i=1:length(c)
```

```
    fprintf('No. %d (length %d)\n\n',i,l(i));
    disp(c{i});
end
fprintf('Transient time: [r0, T] = [%d %d]\n\n',r0,T);
```

References

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