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Extending Newton's law from nonlocal-in-time kinetic energy

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ABSTRACT

We study a new equation of motion derived from a context of classical Newtonian mechanics by replacing the kinetic energy with a form of nonlocal-in-time kinetic energy. It leads to a hypothetical extension of Newton's second law of motion. In a first stage the obtainable solution form is studied by considering an unknown value for the nonlocality time extent. This is done in relation to higher-order Euler–Lagrange equations and a Hamiltonian framework. In a second stage the free particle case and harmonic oscillator case are studied and compared with quantum mechanical results. For a free particle it is shown that the solution form is a superposition of the classical straight line motion and a Fourier series. We discuss the link with quanta interpretations made in Pais–Uhlenbeck oscillators. The discrete nature emerges from the continuous time setting through application of the least action principle. The harmonic oscillator case leads to energy levels that approximately correspond to the quantum harmonic oscillator levels. The solution to the extended Newton equation also admits a quantization of the nonlocality time extent, which is determined by the classical oscillator frequency. The extended equation suggests a new possible way for understanding the relationship between classical and quantum mechanics.

1. Introduction

Historically, Newton's law [1] has taken a central place. However, classical Newtonian mechanics is unable to explain phenomena that are observed at atomic length scales or smaller, where quantum mechanics has been accurate in addressing phenomena of wave–particle duality, quantization, tunnelling, interference, nonlocality and teleportation [2,3]. On the other hand connections between quantum mechanics and classical mechanics have been established in Bohmian mechanics, through the concept of a quantum potential [4]. In relation to stochastic processes the Schrödinger equation has been derived from Newtonian mechanics by Nelson [5]. Nottale outlined a framework of scale relativity with a fractal space–time concept [6] from which a complex valued Newton's law is connected to the Schrödinger equation. Furthermore, the Ehrenfest theorem provides a correspondence principle between the Schrödinger equation and Newton's law in terms of expected values [7]. Other related work on the links between classical mechanics and quantum mechanics have been made e.g. in [8–10]. For explaining the galaxy rotation problem Modified Newtonian dynamics (MOND) [11] has been proposed as a modification to Newton's second law.

Despite the many successes of quantum mechanics, different interpretations on realism of the physical world [12] currently remain possible, which are typically experimentally tested in relation to the Einstein–Podolsky–Rosen paradox [13,14]. Einstein considered the quantum-mechanical description of physical reality to be incomplete, as witnessed also by his "*God doesn't play dice*" statement. In [15] Zurek has characterized the unease with quantum mechanical interpretations as follows: "*Why is a theory that seems to account with precision for everything we can measure still deemed lacking?... at the root of our unease with quantum theory is the clash between the principle of superposition and everyday classical reality in which this principle appears to be violated*". In recent years, the quest for a deterministic theory underlying quantum mechanics has been advocated by 't Hooft [16].

The initial motivation for this Letter originates from a remark made by Feynman in [17, p. 382] on the observability of the kinetic energy functional. Assuming position measurements of coordinate x at successive times $t_{i+1} = t_i + \epsilon$, Feynman noted that the kinetic energy functional cannot be written as $\frac{1}{2}[(x_{k+1} - x_k)/\epsilon]^2$, but can be written to order ϵ as $\frac{1}{2}\frac{(x_{k+1}-x_k)}{\epsilon}\frac{(x_k-x_{k-1})}{\epsilon}$. In other words the position

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differences should be shifted with respect to each other. In this Letter we will take this Feynman's conclusion as a starting point within a different setting. We are curious to learn what is obtained if one would start from an entirely classical, deterministic and corpuscular framework of Newtonian mechanics and plug-in a nonlocal-in-time version of kinetic energy instead of the standard kinetic energy term within this context. Could quantum phenomena emerge? Would it be possible to obtain discreteness within a continuous time setting from an optimality principle?

In order to address these questions, our approach consists of working in two stages. In a first stage, we consider the classical Newtonian framework and insert one additional unknown parameter (the nonlocality time extent) into it by specifying a nonlocal-in-time kinetic energy. As remarked in [18] with respect to quantum mechanics and string theory, adding an element, even in the case when its numerical value is tiny, might drastically change the structure and obtained solution. In fact we are interested here in looking for the least modification to Newtonian mechanics that could result in explaining quantum phenomena. Based on the nonlocal kinetic energy term, we study the high order Euler–Lagrange equations which are shown to result into an extension to Newton's second law of motion. In the second stage, in comparison with quantum mechanics, we study the free particle and harmonic oscillator case to gain further insight into the role of the nonlocality time extent. The free particle case is related to Pais–Uhlenbeck oscillators with interpretation of quanta [19]. For the harmonic oscillator case it turns out that the nonlocality time extent can be quantized and upper bounded in terms of the classical harmonic oscillator frequency.

Related work of nonlocal field theories have been studied in [19–23] and nonlocality with finite time extent in [24,25]. General equations with higher order derivatives were studied in Pais–Uhlenbeck theory [19] with results in connection to anharmonic oscillator problems and quantum mechanics. Higher order derivative models are of interest e.g. in the study of gravity, tachyons and beam theory [26–31]. The resulting Lagrangian that we obtain is singular. In order to connect to the Ostrogradski Hamiltonian [33] formalism it is desirable to start from a regular Lagrangian [24,32]. We consider a regularized Lagrangian which leads to the same limit. For the infinite derivatives case we interpret the work within the existing $(1+1)$-dimensional formalism of nonlocal theory [20–24], which has two time coordinates with one local and one nonlocal coordinate. It can be considered as a generalization to the Ostrogradski formalism for the case of infinite derivative theories. For this Hamiltonian formalism symmetries have been discussed in [21]. A common problem in higher derivative theories is the Ostrogradski instability which leads to the lack of a lower bound on the energy. Several approaches have been explored to avoid the Ostrogradski instability and different interpretations have been given [34–38]. Though the Ostrogradski instability issue also arises in our model and in [19], it can also be avoided in the latter as shown in [39–41].

This Letter is organized as follows. In Section 2 we outline the notion of nonlocal-in-time kinetic energy. In Section 3 we study the related higher order Euler–Lagrange equation for the finite number of derivatives case and discuss the conserved energy. In Section 4 a regularization to the singular Lagrangian is made. This is further connected to the Ostrogradski Hamiltonian. The infinite number of derivatives case is interpreted within the $(1+1)$-dimensional formalism of nonlocal theory. In Section 5 the free particle case is considered with quanta interpretations in view of Pais–Uhlenbeck oscillators. In Section 6 the harmonic oscillator case is discussed, including the quantization of nonlocality time extent.

2. A nonlocal-in-time form of kinetic energy

It is well known that for conservative systems Newton's law of motion $F = m\ddot{x}$ can be obtained from the principle of least action [2] where the equation of motion is obtained as the evolution for which the action $S = \int_{t_0}^{t_f} L\,dt$ is stationary where the integral is over time with time instants t_0, t_f and $\delta x(t_0) = \delta x(t_f) = 0$. $L = T - V$ denotes the Lagrangian containing the kinetic energy term $T = \frac{1}{2}mv^2$ with velocity $v = \dot{x} = \frac{dx(t)}{dt}$, constant mass m and the potential energy term $V(x)$ with point position $x(t)$. The force is $F = -\frac{\partial V}{\partial x}$ for a conservative system and $\ddot{x}$ the acceleration.

Instead of considering this standard notion of kinetic energy, we study here a nonlocal-in-time version of kinetic energy defined as

$$T_\tau = \frac{1}{2}mv(t)\frac{1}{2}\big[v(t+\tau) + v(t-\tau)\big]. \tag{1}$$

Related to this expression for T_τ we take the Taylor approximations

$$\begin{aligned} x(t+\tau) &\simeq x(t) + \tau\dot{x}(t) + \frac{\tau^2}{2!}\ddot{x}(t) + \cdots + \frac{\tau^n}{n!}x^{(n)}(t), \\ x(t-\tau) &\simeq x(t) - \tau\dot{x}(t) + \frac{\tau^2}{2!}\ddot{x}(t) - \cdots + (-1)^n\frac{\tau^n}{n!}x^{(n)}(t) \end{aligned} \tag{2}$$

where $x^{(n)}(t)$ denote the nth order time-derivatives and τ a positive constant. The value of τ is considered to be small relative to the time scale for which the system is studied. For the moment the interpretation of τ is only considered at this abstract mathematical level. The issue of its physical interpretation will be addressed further in Sections 5 and 6.

The nonlocal-in-time kinetic energy based on the nth order Taylor approximations (2) becomes

$$T_{\tau,n} = \frac{1}{2}m\dot{x}\frac{1}{2}\left[\dot{x} + \sum_{k=1}^{n}\frac{\tau^k}{k!}x^{(k+1)} + \dot{x} + \sum_{k=1}^{n}(-1)^k\frac{\tau^k}{k!}x^{(k+1)}\right] = T + \frac{1}{4}m\dot{x}\left[\sum_{k=1}^{n}\big(1+(-1)^k\big)\frac{\tau^k}{k!}x^{(k+1)}\right] \tag{3}$$

with special cases $T_{\tau,1} = T$ and $T_{\tau,2} = \frac{1}{2}m\dot{x}^2 + \frac{1}{4}m\tau^2\dot{x}x^{(3)}$. Correspondingly we denote $L_{\tau,n} = T_{\tau,n} - V$.

3. Higher order Euler–Lagrange equation: Finite *n* case

3.1. Equations of motion

The Lagrangian $L_{\tau,n} = T_{\tau,n} - V$ contains higher order derivatives with $L_{\tau,n}(x, \dot{x}, \ddot{x}, x^{(3)}, \ldots, x^{(N)})$ where $N = n + 1$ denotes the order of the Lagrangian. Note that in relation to a Hamiltonian framework, which will be addressed in the next section, one can consider independent variables $q_i(t)$ such that $q_i = \dot{q}_{i-1}$ for $i = 1, 2, \ldots, N-1$ with $q_0 = x$ and $q = q_0$ [42].

The higher order Euler–Lagrange equation is given by

$$\sum_{j=0}^{N} (-1)^j \frac{d^j}{dt^j} \frac{\partial L_{\tau,n}}{\partial q^{(j)}} = 0 \tag{4}$$

which is the stationary solution to the action $\int_{t_0}^{t_f} L_{\tau,n}\, dt$, under the assumptions that $\delta x^{(j)}(t_0) = \delta x^{(j)}(t_f) = 0$ for $j = 0, 1, \ldots, N-1$ (see e.g. [37]). Furthermore it is assumed that τ is time t independent and $\tau \ll t_f - t_0$.

One has

$$\frac{\partial L_{\tau,n}}{\partial q^{(0)}} = -\frac{\partial V}{\partial q} = F \tag{5}$$

and by denoting $a_k = 1 + (-1)^k$

$$\frac{\partial L_{\tau,n}}{\partial \dot{q}} = m\dot{q} + \frac{1}{4} m \sum_{k=1}^{n} a_k \frac{\tau^k}{k!} q^{(k+1)} \tag{6}$$

which gives

$$\frac{d}{dt} \frac{\partial L_{\tau,n}}{\partial \dot{q}} = m\ddot{q} + \frac{1}{4} m \sum_{k=1}^{n} a_k \frac{\tau^k}{k!} q^{(k+2)}. \tag{7}$$

For the terms $j \geqslant 2$ one has

$$\sum_{j=2}^{N} (-1)^j \frac{d^j}{dt^j} \frac{\partial L_{\tau,n}}{\partial q^{(j)}} = \sum_{k=1}^{N-1} (-1)^{k+1} \frac{d^{k+1}}{dt^{k+1}} \frac{\partial L_{\tau,n}}{\partial q^{(k+1)}} = \sum_{k=1}^{N-1} (-1)^{k+1} \frac{d^{k+1}}{dt^{k+1}} \left(\frac{1}{4} m\dot{q} a_k \frac{\tau^k}{k!} \right) = \sum_{k=1}^{N-1} (-1)^{k+1} \frac{1}{4} m a_k \frac{\tau^k}{k!} q^{(k+2)}. \tag{8}$$

Together this results into

$$F - m\ddot{q} - \frac{1}{4} m \sum_{k=1}^{n} a_k \frac{\tau^k}{k!} q^{(k+2)} + \frac{1}{4} m \sum_{k=1}^{n} (-1)^{k+1} a_k \frac{\tau^k}{k!} q^{(k+2)} = 0 \tag{9}$$

or

$$F - m\ddot{q} - m \sum_{k=1}^{n} r_k \frac{\tau^k}{k!} q^{(k+2)} = 0 \tag{10}$$

where $r_k = \frac{1}{4}(1 - (-1)^{k+1}) a_k$. For k even $r_k = 1$ holds and $r_k = 0$ for k odd.

For n even the Euler–Lagrange equation becomes

$$F = m\ddot{q} + m \sum_{k=1}^{n/2} \frac{\tau^{2k}}{(2k)!} q^{(2k+2)} \tag{11}$$

where all odd derivative terms cancel and only the even derivative terms remain, resulting into reversible motion [19]. For $\tau = 0$ one recovers Newton's second law of motion

$$F = m\ddot{q}. \tag{12}$$

For the case $n = 2$ one obtains

$$F = m\ddot{q} + m \frac{\tau^2}{2} q^{(4)} \tag{13}$$

and for $n = 4$

$$F = m\ddot{q} + m \frac{\tau^2}{2} q^{(4)} + m \frac{\tau^4}{24} q^{(6)}. \tag{14}$$

Making use of the approximations (2) the limit case $n \to \infty$ yields the extended equation

$$F = m \frac{1}{2} \left[\ddot{q}(t + \tau) + \ddot{q}(t - \tau) \right]. \tag{15}$$

Note that a different equation of motion containing advanced and retarded terms was discussed in [43].

3.2. Conserved energy

Given the higher order Lagrangian $L_{\tau,n}$ (not explicitly depending on time), the expression for the conserved energy is given by [44]

$$E=\sum_{k=0}^{N}\sum_{j=0}^{k-1}(-1)^j\left(q^{(k-j)}\frac{d^j}{dt^j}\frac{\partial L_{\tau,n}}{\partial q^{(k)}}\right)-L_{\tau,n}. \tag{16}$$

For the case $n=2$ $(N=3)$ it is straightforward to derive that

$$E_{N=3}=\frac{1}{2}m\dot{q}^2+\frac{1}{2}m\tau^2\dot{q}q^{(3)}-\frac{1}{4}m\tau^2(\ddot{q})^2+V. \tag{17}$$

For the case $n=4$ $(N=5)$ one obtains

$$E_{N=5}=\frac{1}{2}m\dot{q}^2+m\frac{\tau^2}{2!}\dot{q}q^{(3)}+m\frac{\tau^4}{4!}\dot{q}q^{(5)}-m\frac{\tau^4}{4!}\ddot{q}q^{(4)}-\frac{1}{2}m\frac{\tau^2}{2!}(\ddot{q})^2+\frac{1}{2}m\frac{\tau^4}{4!}\left(q^{(3)}\right)^2+V. \tag{18}$$

For $\tau=0$ this specializes to the classically known conserved energy

$$E_{\tau=0}=\frac{1}{2}m\dot{q}^2+V=T+V. \tag{19}$$

4. Hamiltonian framework

4.1. A regularized Lagrangian

For a Lagrangian of order N to be regular the highest order term appearing in the equation of motion should be in $q^{(2N)}$ [24]. Hence the Lagrangian $L_{\tau,n}$ is singular as the highest order term is in $n+2=N+1$. However, a regular Lagrangian can be considered that preserves the same form of the equations, completed with additional terms. This is done by taking the following regularization for n even

$$L^{\mathrm{reg}}_{\tau,n}=L_{\tau,n}+R_{\tau,n} \tag{20}$$

with regularization part

$$R_{\tau,n}=\frac{1}{2}\sum_{j=N-k_N}^{N}(-1)^{j+1}\epsilon_j\left(q^{(j)}\right)^2 \tag{21}$$

where $\epsilon_j=m\frac{\tau^{2j-2}}{(2j-2)!}$ and $k_N=\frac{N-3}{2}$ (a negative k_N value means no regularization part).

Given that $\frac{\partial R_{\tau,n}}{\partial q^{(j)}}=(-1)^{j+1}\epsilon_j q^{(j)}$ the Euler–Lagrange equation contains then the following additional terms

$$\frac{1}{2}\sum_{j=N-k_N}^{N}(-1)^j\frac{d^j}{dt^j}\frac{\partial R_{\tau,n}}{\partial q^{(j)}}=\sum_{j=N-k_N}^{N}(-1)^j(-1)^{j+1}\epsilon_j q^{(2j)}=-\sum_{j=N-k_N}^{N}\epsilon_j q^{(2j)}. \tag{22}$$

The high order Euler–Lagrange equation becomes then

$$F=m\ddot{q}+m\sum_{k=1}^{n/2}\frac{\tau^{2k}}{(2k)!}q^{(2k+2)}+\sum_{j=N-k_N}^{N}\epsilon_j q^{(2j)}=m\ddot{q}+m\sum_{k=2}^{N}\frac{\tau^{2k-2}}{(2k-2)!}q^{(2k)} \tag{23}$$

which leads to the same limit (15) for $n\to\infty$.

For the case $n=2$ $(N=3,\ k_N=0)$ this gives

$$L^{\mathrm{reg}}_{\tau,2}=\frac{1}{2}m\dot{q}^2+\frac{1}{2}m\frac{\tau^2}{2!}\dot{q}q^{(3)}-V(q)+\frac{1}{2}\epsilon_3\left(q^{(3)}\right)^2 \tag{24}$$

with equation of motion

$$F=m\ddot{q}+m\frac{\tau^2}{2}q^{(4)}+\epsilon_3 q^{(6)} \tag{25}$$

with $\epsilon_3=m\frac{\tau^4}{4!}$. For $L^{\mathrm{reg}}_{\tau,2}$ the conserved energy (16) equals (18) with $E^{\mathrm{reg}}_{N=3}=E_{N=5}$.

For the case $n=4$ $(N=5,\ k_N=1)$ one obtains

$$L^{\mathrm{reg}}_{\tau,4}=\frac{1}{2}m\dot{q}^2+\frac{1}{2}m\frac{\tau^2}{2!}\dot{q}q^{(3)}+\frac{1}{2}m\frac{\tau^3}{3!}\dot{q}q^{(5)}-V(q)-\frac{1}{2}\epsilon_4\left(q^{(4)}\right)^2+\frac{1}{2}\epsilon_5\left(q^{(5)}\right)^2 \tag{26}$$

with equation of motion

$$F=m\ddot{q}+m\frac{\tau^2}{2!}q^{(4)}+m\frac{\tau^4}{4!}q^{(6)}+\epsilon_4 q^{(8)}+\epsilon_5 q^{(10)} \tag{27}$$

with $\epsilon_4=m\frac{\tau^6}{6!}$, $\epsilon_5=m\frac{\tau^8}{8!}$. In the limit case $n\to\infty$ one also obtains (15), based now on the regular Lagrangian $L^{\mathrm{reg}}_{\tau,n}$.

4.2. Ostrogradski Hamiltonian

The regularized Lagrangian (20) is non-degenerate. One can define then N coordinates and N conjugate momenta in the canonical phase space in the following way [24,33]

$$Q_I = q^{(I-1)}, \qquad P_I = \sum_{J=I}^{N} \left(-\frac{d}{dt}\right)^{J-I} \frac{\partial L_{\tau,n}^{\text{reg}}}{\partial q^{(J)}}. \tag{28}$$

The Hamiltonian is then given by

$$H = \sum_{I=1}^{N} P_I \dot{Q}_I - L_{\tau,n}^{\text{reg}} \tag{29}$$

with canonical equations

$$\dot{Q}_I = \frac{\partial H}{\partial P_I}, \qquad \dot{P}_I = -\frac{\partial H}{\partial Q_I}. \tag{30}$$

For the case $N = 3$ it is straightforward to verify that $H = E_{N=3}^{\text{reg}} = E_{N=5}$ is conserved.

4.3. $(1+1)$-dimensional formalism of nonlocal theory

As discussed in the formalism by Gomis et al. [20–23] and Woodard [24], nonlocal theories are described by actions that contain an infinite number of temporal derivatives. For such theories there exists an equivalent formulation in a space–time of one dimension higher. Based on this formalism a $1+1$ field theory can be considered having two time coordinates t and λ. A Hamiltonian formalism can be constructed then in such a way that the evolution is local with respect to one of these two coordinates (local in t and nonlocal in λ). We briefly review here a number of results that are relevant in the context of this Letter.

For $n \to \infty$ one has the nonlocal Lagrangian $L^{\text{non}} = L_{\tau,\infty}^{\text{reg}}$ and the action $S[q] = \int L^{\text{non}}(t)\,dt$. The higher order theory can be embedded in a nonlocal setting having an infinite dimensional phase space with Taylor basis [20]

$$Q(t,\lambda) = \sum_{k=0}^{\infty} e_k(\lambda) q^k(t), \qquad P(t,\lambda) = \sum_{k=0}^{\infty} e^k(\lambda) p_k(t) \tag{31}$$

with $P(t,\lambda)$ the canonical momentum of $Q(t,\lambda)$ and e_k, e^k an orthonormal basis satisfying the properties $e_k(\lambda) = \lambda^k/k!$, $e^k(\lambda) = (-\partial_\lambda)^k \delta(\lambda)$, $\int e^k(\lambda) e_l(\lambda)\,d\lambda = \delta_l^k$, $\sum_{k=0}^{\infty} e^k(\lambda) e_k(\lambda') = \delta(\lambda - \lambda')$. For the dynamical variables $Q(t,\lambda)$ one has $Q(t,\lambda) = q(\lambda + t)$ for which $\dot{Q}(t,\lambda) = \frac{\partial}{\partial\lambda} Q(t,\lambda)$. The coefficients in (31) are then new canonical variables with Poisson brackets $\{Q(t,\lambda), P(t,\lambda')\} = \delta(\lambda - \lambda')$, $\{q^k(t), p_l(t)\} = \delta_l^k$.

The Hamiltonian becomes

$$H\big(t,[Q,P]\big) = \int \left[P(t,\lambda) \frac{\partial}{\partial\lambda} Q(t,\lambda) - \delta(\lambda)\mathcal{L}(t,\lambda) \right] d\lambda \tag{32}$$

and

$$H = \sum_{k=0}^{\infty} p_k(t) q^{k+1}(t) - L^{\text{non}}\big(q^0, q^1, q^2, \ldots\big). \tag{33}$$

The Lagrangian density $\mathcal{L}(t,\lambda)$ is constructed from the original nonlocal Lagrangian L^{non} by replacing $q(t) \to Q(t,\lambda)$ and $\frac{d^j}{dt^j} q(t) \to \frac{d^j}{d\lambda^j} Q(t,\lambda)$. Furthermore one replaces $q(t+\tau) \to Q(t,\lambda+\tau)$. The formalism can be viewed as a generalization of the Ostrogradski formalism for the case of infinite derivative theories.

The Hamiltonian equations are given by [20–23]

$$\dot{Q}(t,\lambda) = \frac{\partial}{\partial\lambda} Q(t,\lambda), \qquad \dot{P}(t,\lambda) = \frac{\partial}{\partial\lambda} P(t,\lambda) + \mathcal{E}(t;0,\lambda) \tag{34}$$

with functional derivative $\mathcal{E}(t;\lambda',\lambda) = \frac{\delta \mathcal{L}(t,\lambda')}{\delta Q(t,\lambda)}$.

In this formalism, the solutions to these $(1+1)$-dimensional field equations are related to the Euler–Lagrange equations of the original nonlocal Lagrangian if a constraint on the momentum is imposed

$$\varphi(t,\lambda) = P(t,\lambda) - \int \frac{\epsilon(\lambda) - \epsilon(\lambda')}{2} \mathcal{E}(t;\lambda',\lambda)\,d\lambda' \approx 0 \tag{35}$$

with $\approx$ denoting that the equations hold on the constraint surface (weak equality) and $\epsilon(\lambda)$ the sign distribution. Also stability of this constraint is imposed $\dot{\varphi}(t,\lambda) \approx 0$. Further consistency conditions give an infinite set of Hamiltonian constraints

$$\psi(t,\lambda) = \int \mathcal{E}(t;\lambda',\lambda)\,d\lambda' \approx 0 \quad \text{for } -\infty < \lambda < \infty \tag{36}$$

corresponding to the Euler–Lagrange equation. The constraints (35), (36) belong to the second class of Dirac constraints [45].

According to [21] the symmetry generator in the Hamiltonian formalism is

$$G(t) = \int \left[P(t,\lambda)\delta Q(t,\lambda) - \delta(\lambda)\mathcal{K}(t,\lambda) \right] d\lambda \tag{37}$$

where $\delta Q(t,\lambda)$ and $\mathcal{K}(t,\lambda)$ are constructed from $\delta q(t)$ and $k(t)$ respectively (where for the nonlocal Lagrangian $\delta L^{\text{non}}(t) = \frac{d}{dt}k(t)$) by the same replacement rules as before. The generator $G(t)$ generates the transformation of $Q(t,\lambda)$ such that $\delta Q(t,\lambda) = \{Q(t,\lambda), G(t)\}$. The following properties are then satisfied [21]: $G(t)$ is a conserved quantity where

$$\frac{d}{dt}G(t) = \left\{G(t), H(t)\right\} + \frac{\partial}{\partial t}G(t) = 0. \tag{38}$$

All the constraints are invariant under the symmetry transformations

$$\left\{\varphi(t,\lambda), G(t)\right\} \approx 0, \qquad \left\{\psi(t,\lambda), G(t)\right\} \approx 0. \tag{39}$$

Furthermore the Hamiltonian (32) is the generator of time translations.

5. Free particle case

5.1. Quanta

Let us now consider the free particle case ($F = 0$) for the extended equation (11)

$$m\ddot{q} + m\sum_{k=1}^{n/2} \frac{\tau^{2k}}{(2k)!} q^{(2k+2)} = 0. \tag{40}$$

Proposing a solution of the form $q(t) = e^{i\omega t}$ gives

$$m\omega^2 \left(1 - \frac{\tau^2}{2!}\omega^2 + \frac{\tau^4}{4!}\omega^4 - \frac{\tau^6}{6!}\omega^6 + \cdots\right) e^{i\omega t} = 0 \tag{41}$$

which is a finite n approximation to

$$m\omega^2 \cos(\tau\omega) e^{i\omega t} = 0. \tag{42}$$

The latter equation is satisfied for $\omega_l = 0$ (with multiplicity 2) or $\omega_l = \frac{2l+1}{2}\frac{\pi}{\tau}$ with $l \in \mathbb{Z}$. As a result the solution is of the form $q(t) = c_1 + c_2 t + \sum_l a_{1,l}\cos(\omega_l t) + \sum_l a_{2,l}\sin(\omega_l t)$ with unknown coefficients $c_1, c_2, a_{1,l}, a_{2,l}$. Here the oscillatory modes (expressed by a Fourier series) are superimposed on the classical straight line motion. Since a Fourier series can reproduce any piecewise continuous bounded function on the interval $[t_0, t_f]$ the solution form encompasses all piecewise continuous, bounded, monotonic in time paths connecting the initial and final points.

Starting from (15)

$$m\frac{1}{2}\left[\ddot{q}(t+\tau) + \ddot{q}(t-\tau)\right] = 0 \tag{43}$$

leads to the same result: let us propose a solution of the form $q(t+\tau) = A(t)B(\tau)$ with $A(t) = e^{i\omega t}$ and $B(\tau) = e^{i\omega\tau}$ such that $q(t) = A(t)B(0)$ with $B(0) = 1$. One obtains then

$$m\omega^2 \frac{1}{2}\left[B(\tau) + B(-\tau)\right]A(t) = 0 \tag{44}$$

with $\ddot{A}(t) = -\omega^2 A(t)$ and $\frac{1}{2}[B(\tau) + B(-\tau)] = \cosh(i\tau\omega) = \cos(\tau\omega)$. For the mode $\omega_l = 0$ the solution is then characterized by $\ddot{A}(t) = 0$ which gives the classical straight line motion. The non-zero ω_l modes lead to classical harmonic oscillator solutions $\ddot{A}(t) = -\omega_l^2 A(t)$.

The model can also be written as

$$mD^2 \cosh(\tau D) q(t) = 0 \tag{45}$$

with D the time derivative operator. According to Pais and Uhlenbeck [19] the following product representation exists

$$\cosh(\tau D) = \prod_{l=1}^{\infty}\left(1 + \frac{D^2}{\omega_l^2}\right) \tag{46}$$

where the factors correspond to quanta.

A correspondence with quantum mechanical energy levels is obtained as follows. Defining $E_l^+ = \frac{2l+1}{2}\hbar\omega_*$, $l \in \mathbb{N}_0$ with $\omega_* = \pi/\tau$ and $E_0^+ = \frac{1}{2}\hbar\omega_*$, the quantum mechanical energy levels relate as

$$E_{\text{qm},l} = E_l^+ - E_0^+ = l\hbar\omega_*, \quad l \in \mathbb{N}_0 \tag{47}$$

with zero-point energy $\hbar\omega_*$. Alternatively, this is also obtained by $E_{\text{qm},l} = -E_l^- + E_0^-$ where $E_l^- = \frac{2l-1}{2}\hbar\omega_*$, $-l \in \mathbb{N}_0$ and $E_0^- = -\frac{1}{2}\hbar\omega_*$.

The negative ω_l values lead however to a negative energy due the Ostrogradski instability problem, suffering from the ghost problem. Also the models studied by Pais and Uhlenbeck [19] and 't Hooft [16,36,46] face this problem. On the other hand, several approaches have been explored to avoid the Ostrogradski instability and different interpretations have been given [34–41]. The reality condition on the Fourier modes of [34, Eq. (3.9)] is satisfied which reduces the physical degrees of freedom by a factor two.

For the physical interpretation of τ in the free particle case, one can relate τ to the Einstein–de Broglie relation $m_0c^2 = h\nu_B$ in a rest frame and setting $\omega_B = 2\pi\nu_B = \pi/\tau$ where m_0 is the rest mass of the particle and c the speed of light [53]. For the existence of an internal clock frequency hypothesized by de Broglie, experimental evidence has recently been found in [54].

5.2. Conserved momentum

In the free particle case the following extended momentum is conserved based on the expression for $L_{\tau,n}$ in (11) and n even

$$p_\tau = m\dot{q} + m\sum_{k=1}^{n/2}\frac{\tau^{2k}}{(2k)!}q^{(2k+1)} \tag{48}$$

which is a finite n approximation to

$$p_\tau = m\frac{1}{2}\big[\dot{q}(t+\tau) + \dot{q}(t-\tau)\big]. \tag{49}$$

For $\tau = 0$ this gives the classical momentum $p = m\dot{q}$. For the regularized Lagrangian an expression of the same form is obtained but with additional terms in the expansion, according to (23).

5.3. Initial value problem

For the case $n = 2$ in (11), proposing a solution of the form $q(t) = e^{i\omega t}$ gives $\omega = 0$, $\omega = \pm\sqrt{2}/\tau$ where $\omega = 0$ with multiplicity 2 corresponds to the straight line component. For a given initial condition $q(0) = q_0$, $\dot{q}(0) = \dot{q}_0$, $\ddot{q}(0) = \ddot{q}_0$, $\dddot{q}(0) = \dddot{q}_0$ one gets then the coefficients

$$c_1 = q_0 + \frac{\ddot{q}_0}{\omega^2}, \qquad c_2 = \dot{q}_0 + \frac{\dddot{q}_0}{\omega^2}, \qquad a_1 = -\frac{\ddot{q}_0}{\omega^2}, \qquad a_2 = -\frac{\dddot{q}_0}{\omega^3}. \tag{50}$$

Hence, the larger a_1, a_2 in absolute value, the more the solution will deviate from a straight line motion. Such finite n approximations are at best qualitative approximations to the solution form of the infinite derivatives case, also because for the infinite derivatives case (15) the specification of initial values becomes different. In general this is a non-trivial issue [34]. In [26] it was argued that one rather has to find then initial conditions that are consistent with the equation. In the $(1+1)$-dimensional formalism [20] an initial line segment has to be given. In [34] it has been explained that differential equations with infinitely many derivatives, do not necessarily need an infinite number of conditions for the initial value specification.

In the free particle case it can be verified that

$$q(t+\tau) = c_1 + c_2 t - q(t-\tau) \tag{51}$$

is a solution to (43). Related but different retarded equations have also been studied in [48,49]. A major difference is that the finite time extent τ enters the acceleration instead of the position variable in our model.

Note that for an initial value problem that admits $c_1 = c_2 = 0$ one can write

$$\begin{bmatrix} q(t+\tau) \\ q(t) \end{bmatrix} = U \begin{bmatrix} q(t) \\ q(t-\tau) \end{bmatrix} \quad \text{with } U = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{52}$$

where U is a unitary matrix. This can be interpreted as mapping path segments (of time extent τ) over time t. The equation $q(t+\tau) + q(t-\tau) = 0$ expresses then a periodicity property of the solution which also needs to be compatible with the initial condition. For differential–difference equations it is known that while to problem appears to be infinite dimensional, it can occur that the underlying problem is finite dimensional [47]. Eq. (52) represents the problem in a second order difference equation form (when considering discrete time instants $t = j\tau$ for $j \in \mathbb{N}$).

An example with non-zero c_2 value is illustrated in Fig. 1 with initial value specification $q(t) = 0$ for $t \in [-\tau, 0]$ and $c_1 = 0, c_2 = 1$.

5.4. Impulse response

For the case of an infinite number of derivatives we study the impulse response of the system for zero initial state vector and $m = 1$. One obtains the transfer function

$$H(s) = \frac{1}{s^2\cosh(\tau s)} \tag{53}$$

with $s = \sigma + i\omega$ denoting a complex number and $\cosh(\tau s) = 1 + \frac{\tau^2}{2!}s^2 + \frac{\tau^4}{4!}s^4 + \cdots$. The inverse Laplace transform $\mathsf{L}^{-1}\{H(s)\} = h(t)$ gives then the impulse response [50,51]

$$h(t) = \int_0^t f(u)\,du \tag{54}$$

with

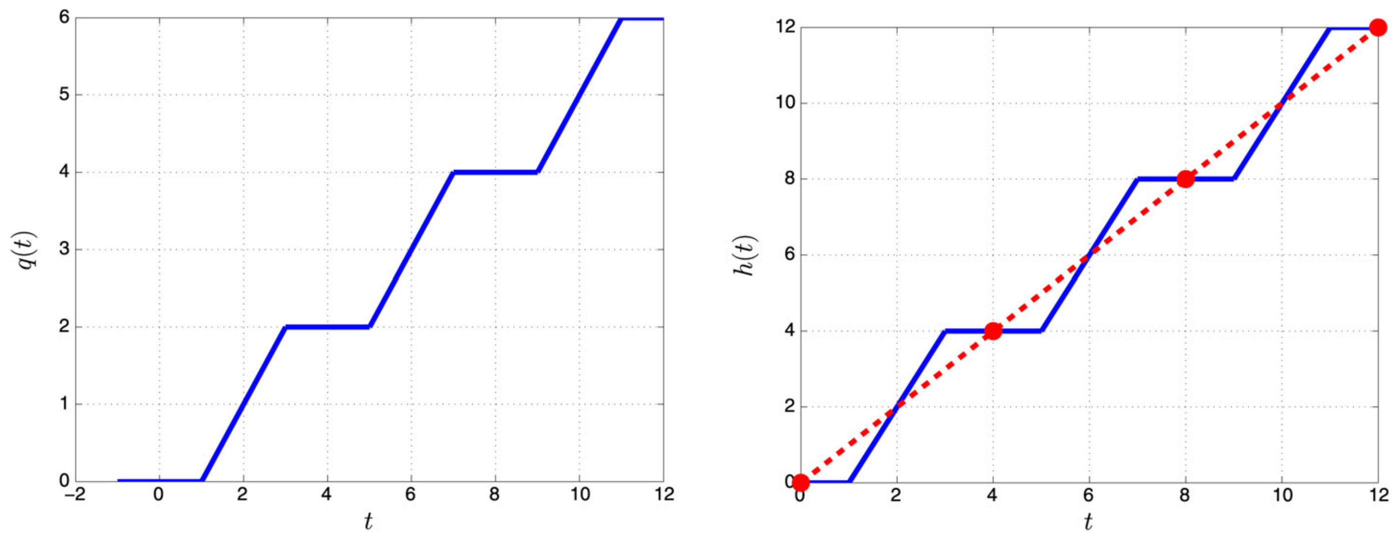

Fig. 1. (Left) Solution to (51) for initial value specification $q(t) = 0$ for $t \in [-\tau, 0]$ and $c_1 = 0, c_2 = 1$; (Right) impulse response $h(t)$ for a free particle for $n \to \infty$ and zero initial condition. The motion according to this extended equation is a zigzag path (solid line) around the classical Newtonian straight line solution (dashed line). $\tau = 1$ is set in both subfigures.

$$f(t) = 2\sum_{j=0}^{\infty}(-1)^j \operatorname{step}\bigl(t-(2j+1)\tau\bigr) \tag{55}$$

where $\operatorname{step}(t-a)$ denotes the Heaviside unit step at time a with $\operatorname{step}(t-a) = 0$ if $t \leqslant a$ and $\operatorname{step}(t-a) = 1$ if $t > a$. We make use of the property that $\mathsf{L}^{-1}\{F(s)/s\} = \int_0^t f(u)\,du$ with Laplace transform $\mathsf{L}\{f(t)\} = F(s)$ where $F(s) = 1/(s\cosh(\tau s))$ and $H(s) = F(s)/s$. The free particle follows a zigzag path in space–time around the Newton path (Fig. 1). At discrete time instants $t = (2j+1)\tau$ $(j = 0, 1, 2, 3, \ldots)$ a discontinuous jump takes place in the velocity. If one would observe this path only at the discrete time instants $t = 4j\tau$ $(j = 0, 1, 2, 3, \ldots)$ it coincides with the sampled version of the Newton straight line solution of the impulse response. A related transfer function $1/\cosh(\tau s)$ has been studied by Pais and Uhlenbeck in [19].

6. Harmonic oscillator case

6.1. Threshold frequency

For the harmonic oscillator case with $V(q) = \frac{1}{2}\kappa q^2$ for $\kappa > 0$, starting from (15) one obtains

$$m\frac{1}{2}\bigl[\ddot{q}(t+\tau) + \ddot{q}(t-\tau)\bigr] + \kappa q(t) = 0. \tag{56}$$

We propose a solution of the form $q(t+\tau) = A(t)B(\tau)$ with $A(t) = e^{i\omega t}$ and $B(\tau) = e^{i\omega\tau}$ such that

$$\left(-m\omega^2\frac{1}{2}\bigl[B(\tau) + B(-\tau)\bigr] + \kappa B(0)\right)A(t) = 0. \tag{57}$$

For the candidate oscillatory modes, ω_l are the solutions to the equation

$$\omega^2\cos(\tau\omega) = \frac{\kappa}{m}. \tag{58}$$

It follows that for the case $\tau = 0$ the classical harmonic oscillator result $\omega^2 = \frac{\kappa}{m}$ is recovered. The property

$$\omega^2 \geqslant \frac{\kappa}{m} \tag{59}$$

holds such that there exists a threshold frequency like in the photoelectric effect [52]. A large amount of additional eigenfrequencies exist in the system in comparison with the classical harmonic oscillator case.

Let us rewrite now (58) in terms of an unknown α where $\omega = \alpha\omega_c$ with $\omega_c^2 = \frac{\kappa}{m}$ known. From (59) one obtains then the constraint $\alpha^2 \geqslant 1$. This leads to the condition

$$\cos(\tau\omega_c\alpha) = \frac{1}{\alpha^2}, \quad \alpha^2 \geqslant 1. \tag{60}$$

6.2. Energy levels

For α sufficiently large, the solution can be approximated by

$$\cos(\tau\omega_c\alpha) \simeq 0 \tag{61}$$

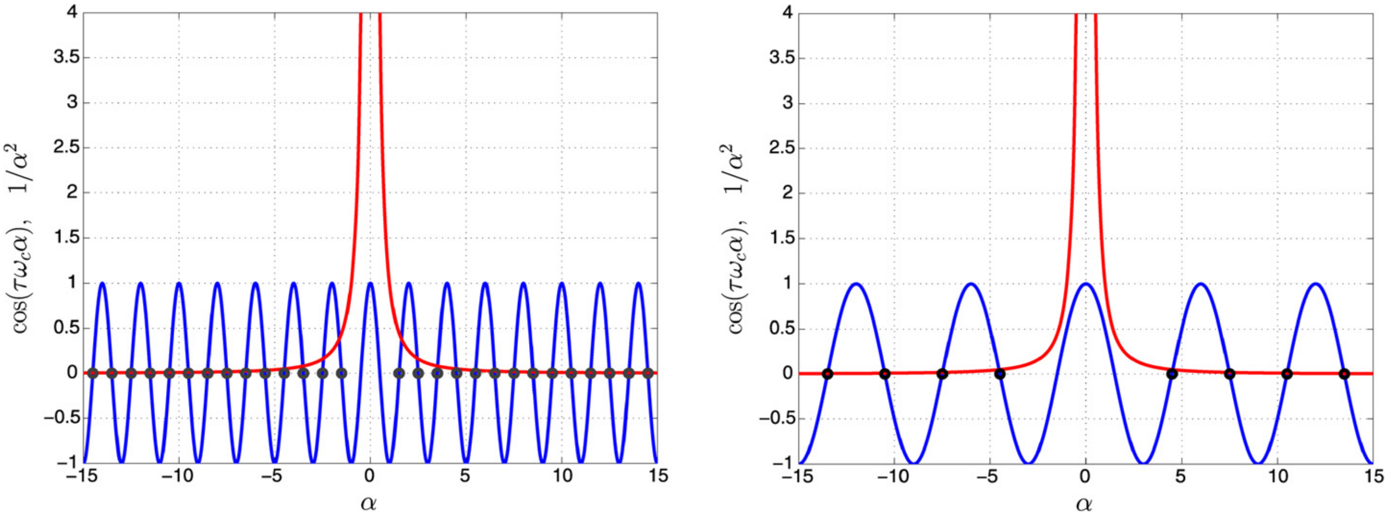

Fig. 2. Harmonic oscillator case. The intersection of the curve $\omega^2\cos(\tau\omega_c\alpha)$ (blue) as a function of α with the curve $1/\alpha^2$ (red) gives candidate oscillatory modes for the system. This intersection gives an approximation to the values $\alpha = \frac{2l+1}{2}(2r+1)$ which are indicated by the black circles. (Left) $r = 0$; (Right) $r = 1$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this Letter.)

where the approximation accuracy is better for increasing α. This is illustrated in Fig. 2.

Note that this approximation also becomes better for a smaller value of $\tau\omega_c$. Let us control this factor by proposing that $\tau\omega_c = \pi/(2r+1)$ where $r \in \mathbb{N}_\infty$ denotes a finite natural number. Checking under which conditions this can be a solution to (61) gives

$$\tau\omega_c\alpha = \frac{1}{2r+1}\pi\alpha = \frac{2l+1}{2}\pi \tag{62}$$

with $l \in \mathbb{Z}_{0,-1}$. The values $l = 0, -1$ are excluded here because for $r = 0$ it would violate the condition $\alpha^2 \geqslant 1$. This yields then the following possible α values

$$\alpha = \frac{2l+1}{2}(2r+1), \quad l \in \mathbb{Z}_{0,-1}, \ r \in \mathbb{N}_\infty \tag{63}$$

where r is an additional quantum number. Note that $\alpha = \frac{2l+1}{2}$ constitutes the same set of values, but leads to a different subset in terms of differences between subsequent energy levels. The quantum number r has the same effect as applying $2r+1$ times the raising operator in quantum mechanics [3].

For $r = 0$ the lowest α value (in absolute value) equals 3/2. On the other hand, for r large one obtains an intersection at $\alpha = 1$ giving the solution $\omega = \omega_c$. This ground state frequency property is consistent with [55].

The two quantum numbers generate partially coinciding α values. A non-redundant set is given by $\{\ldots, -\frac{9}{2}, -\frac{7}{2}, -\frac{5}{2}, -\frac{3}{2}, -1, 1, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \ldots\}$. A correspondence with the energy levels of a quantum harmonic oscillator is obtained as follows. Defining $E_l^+ = \frac{2l+1}{2}\hbar\omega_c$, $l \in \mathbb{N}_0$ and $E_c^+ = \hbar\omega_c$, the quantum mechanical energy levels relate as

$$E_{\mathrm{qm},l} = E_l^+ - E_c^+ = \left(l + \frac{1}{2}\right)\hbar\omega_c, \quad l \in \mathbb{N} \tag{64}$$

with zero-point energy $\frac{1}{2}\hbar\omega_c$. Alternatively, this is also obtained by $E_{\mathrm{qm},l} = -E_l^- + E_c^-$ where $E_l^- = \frac{2l-1}{2}\hbar\omega_c$, $-l \in \mathbb{N}_0$ and $E_c^- = -\hbar\omega_c$.

6.3. *Quantization of* τ

In general, one obtains from (63) that τ is quantized as

$$\tau = \frac{\pi}{(2r+1)\omega_c}, \quad r \in \mathbb{N}_\infty. \tag{65}$$

As a consequence there exists an upper bound on this nonlocality time extent:

$$2\tau \leqslant \frac{\pi}{\nu_c} \tag{66}$$

which is determined by the classical harmonic oscillator frequency $\omega_c = 2\pi\nu_c$. The classical harmonic oscillator is obtained for $\tau = 0$ which corresponds to $\alpha = 1$.

The solution form can be interpreted as a superposition $q(t) = \sum_r \eta_r q_r(t)$ where for each subproblem with solution $q_r(t)$ the τ value is known from $\tau = \frac{\pi}{(2r+1)\omega_c}$:

$$m\frac{1}{2}\left[\ddot{q}_r\left(t + \frac{\pi}{(2r+1)\omega_c}\right) + \ddot{q}_r\left(t - \frac{\pi}{(2r+1)\omega_c}\right)\right] + \kappa q(t) = 0, \quad r \in \mathbb{N} \tag{67}$$

where $r = \infty$ corresponds to the classical harmonic oscillator with $\tau = 0$.

In view of the results of the harmonic oscillator case and its striking similarity with the free particle case, theoretically one can also introduce the additional quantum number r into the free particle condition $\omega^2 \cos(\tau\omega) = 0$ by treating $\omega = \alpha\omega_B$ relative to the de Broglie frequency ω_B and imposing $\tau\omega_B = \pi/(2r+1)$ with $r \in \mathbb{N}_\infty$. Then a similar role is played by ω_B in the free particle case as ω_c in the harmonic oscillator case. That also leads to a τ quantization in the free particle case with nonlocality time extent $2\tau \leqslant \pi/\nu_B$.

7. Conclusion

A hypothetical model has been proposed with an extension to Newton's second law of motion. Though the study of the free particle case and harmonic oscillator case illustrate that the extension is able to explain certain quantum phenomena of quantization and discrete nature, further studies are needed to investigate whether e.g. tunneling and interference might be explainable by this model or not. Also links with relativistic quantum mechanics might be established with respect to Schrödinger's *Zitterbewegung* [7] and de Broglie waves in the effect of clock desynchronization [56]. Finally, we have shown that the nonlocality finite time extent can be expressed in terms of classically known quantities for the harmonic oscillator case. This suggests that when taking this extended Newton equation approach, possible sources of non-determinism should be attributed to the level of an initial value or boundary value specification.

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