The Dirac Delta: Properties and Representations

Concepts of primary interest:

- **Sequences of functions**
- **Multiple representations**
- **Formal properties**
- **Dirac deltas in 2 and 3 dimensions**
- **Dirac deltas in generalized ortho-normal coordinates**
- **Green Function for the Laplacian**

Examples:

- **Multiple zeroes of the argument**
- **Endpoint zeroes of the argument**
- **Green functions -- see Tools of the Trade**

Mega-Application

- **Green function for the Laplace operator**

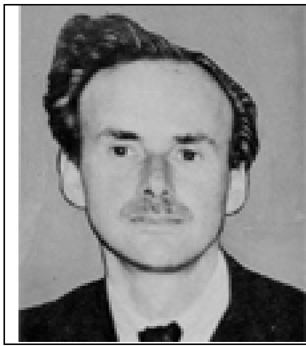
**** Use ${}^{1}D_n(x)$ to introduce the delta and its properties.

*** Change the dimensions to the inverse of the dimension of the integration variable

**** Add vanhoys little delta perturbation at the center of a square well.

Continuous mass and charge distributions are common in physics. Often, as models, point charges and point masses need to be combined with continuous distributions of mass or charge. The Dirac delta *function* is introduced to represent a finite chunk packed into a zero width bin or into zero volume. To begin, the defining formal properties of the Dirac delta are presented. A few applications are presented near the end of this handout. The most significant example is the identification of the Green function for the Laplace problem with its applications to electrostatics.

Contact: tank@alumni.rice.edu



Dirac, P(aul). A. M. (1902-1984) English physicist whose calculations predicted that particles should exist with negative energies. This led him to suggest that the electron had an "antiparticle." This antielectron was discovered subsequently by Carl Anderson in 1932, and came to be called the positron. Dirac also developed a tensor version of the Schrödinger equation, known as the Dirac equation, which is relativistically correct. For his work on antiparticles and wave mechanics, he received the Nobel Prize in physics in 1933.

http://scienceworld.wolfram.com/biography/Dirac.html

Defining Property: The Dirac delta ***function*** $\delta(x - x_0)$ is defined by the values of its integral.

$$\int_a^b \delta(x - x_0)\, dx = \begin{cases} 1 \text{ if } & x_0 \in (a,b) \\ 0 \text{ if } & x_0 \notin [a,b] \end{cases} \quad \text{and } \delta(x - x_0) = 0 \text{ for } x \neq x_0 \qquad \text{[DD.1]}$$

<u>where the integration limits run in the positive sense</u> (b > a). It follows that:

$$\int_a^b f(x)\, \delta(x - x_0)\, dx = \begin{cases} f(x_0) \text{ if } & x_0 \in (a,b) \\ 0 \quad \text{if} & x_0 \notin [a,b] \end{cases} \qquad \text{[DD.2]}$$

for any function $f(x)$ that is continuous at x_o.

NOTE: The defining properties require that the integration limits run in the positive sense. (b > a)

Comparison of the Dirac and Kronecker Deltas

In a sum, the Kronecker delta δ_{km} is defined by its action in sums over an integer index.

$$\sum_{k=k_{lower}}^{k_{upper}} f(k)\, \delta_{km} = \begin{cases} f(m) \text{ if } & k_{lower} \leq m \leq k_{upper} \\ 0 \quad \text{if } & m \notin [k_{lower}, k_{upper}] \end{cases}$$

When the terms of a sum over integers contain a Kronecker delta as a factor, the action of summing over a range of integers k by steps of 1 is to yield a result equal to the value of the one term for which $k = m$ with Kronecker evaluated as one. That is: the entire sum over k evaluates to the one term in which the summation free index is equal to m, the other index of the Kronecker delta. This action is equivalent to the definition that $\delta_{km} = 1$ for $k = m$ and $\delta_{km} = 0$ for $k \neq m$.

The Dirac delta ***function*** $\delta(x - x_0)$ is defined by its action (the sifting property).

$$\int_a^b f(x)\, \delta(x - x_0)\, dx = \begin{cases} f(x_0) \text{ if } & x_0 \in (a,b) \\ 0 \quad \text{if } & x_0 \notin [a,b] \end{cases}$$

When an integrand contains a Dirac delta as a factor, the action of integrating in the positive sense over a region containing a zero of the delta's argument is to yield a result equal to the rest of the integrand evaluated for the value of the free variable *x* that makes the argument of the Dirac delta vanish. This action is equivalent to the definition that $\delta(x - x_o)$, the Dirac delta, is a function that has an area under its curve of 1 for any interval containing x_o and that is zero for $x \neq x_o$.

Derivative Property: Integration by parts, establishes the identity:

$$\int_a^b f(x)\, \frac{d}{dx}\left[\delta(x-x_0)\right]\, dx = \begin{cases} -\left.\dfrac{df(x)}{dx}\right|_{x=x_0} & if \;\; x_0 \in (a,b) \\ 0 \quad if \quad x_0 \notin [a,b] \end{cases} \qquad \text{[DD.3]}$$

Use integration by parts:

proof: $$\int_a^b f(x)\, \frac{d}{dx}\left[\delta(x-x_0)\right]\, dx = f(x)\, \delta(x-x_0)\Big|_a^b - \int_a^b \left[\frac{d}{dx}\, f(x)\right] \delta(x-x_0)\, dx$$

Recall that $\delta(b-x_0)$ = 0 and $\delta(a-x_0)$ = 0 as

$b - x_o \neq 0$ and $a - x_o \neq 0$ given that $a < x_o < b$.

Even Property: The Dirac delta acts as an even function.

The change the integration variable $u = -(x - x_o)$ quickly establishes the even property:

$$\int_a^b f(x)\, \delta(-[x-x_0])\, dx = \int_{x_0-a}^{x_0-b} f(x_0-u)\, \delta(u)\, (-du) = \int_{x_0-b}^{x_0-a} f(x_0-u)\, \delta(u)\, (+du)$$

$$= \int_{x_0-b}^{x_0-a} f(x_0-u)\, \delta(u)\, (+du) = \begin{cases} f(x_0) \;\; if \;\; x_0 \in (a,b) \\ 0 \quad if \;\; x_0 \notin [a,b] \end{cases}$$

$$\int_a^b f(x)\, \delta(-[x-x_0])\, dx \equiv \int_a^b f(x)\, \delta(+[x-x_0])\, dx = f(x_0)$$

Note that the condition that $b > a$ ensures that $(x_0 - a) > (x_0 - b)$. That is: the integration limits run in the positive sense.

Scaling Property: The final basic identity involves scaling the argument of the Dirac delta. A change of integration variable $u = k\, x$ quickly establishes that:

$$\int_a^b f(x)\, \delta(k\, [x-x_0])\, dx = \begin{cases} \dfrac{f(x_0)}{|k|} & if \;\; x_0 \in (a,b) \\ 0 & if \;\; x_0 \notin [a,b] \end{cases} \qquad \text{[DD.4]}$$

$$\int_a^b f(x)\ \delta(k\ [x - x_0])\ dx =$$

$$\int_{ka}^{kb} f([u/k])\ \delta(u - k\,x_0)\ (1/k)\ du = \begin{cases} (1/|k|)\quad f([k\,x_0/k]) & if\ \ k\,x_0 \in (k\,a, k\,b) \\ 0 & if\ \ k\,x_0 \notin [k\,a, k\,b] \end{cases}$$

Note: If $k < 0$, the limits of the integral run in the negative sense after the change of variable. Returning the limits to the positive sense is equivalent to dividing by $|k|$ rather than by k.

$$\int_{ka}^{kb} f([u/k])\ \delta(u - k\,x_0)\ (1/k)\ du = (1/|k|)\int_{u_<}^{u_>} f([u/k])\ \delta(u - k\,x_0)\ du$$

Advanced Scaling Property: The advanced scaling applies to a Dirac deltas with a function as its argument. As always, the functions $f(x)$ and $g(x)$ are continuous and continuously differentiable.

$$\int_a^b f(x)\ \delta\left[g(x) - g(x_0)\right]\ dx = \begin{cases} f(x_0)/\left|\frac{dg}{dx}\right|\Big|_{x=x_0} & if\ \ x_0 \in (a,b) \\ 0 & if\ \ \ x_0 \notin [a,b] \end{cases} \qquad \text{[DD.5]}$$

Using the absolute value $|\frac{dg}{dx}|$ is equivalent to returning the limits to positive order in the local of the argument zero after a change of variable in that case that $\frac{dg}{dx} < 0$. Clearly functions $g(x)$) with first order zeroes are to be used. If $g(x)$ has a second order zero ($g(x_o) = 0$ and $\frac{dg}{dx} = 0$ at x_o), the expression is *undefined*. The advanced scaling property is to be established in a problem, but it can be motivated by approximating the delta's argument around each zero using a Taylor's series as:

$g(x) - g(x_0) \approx \frac{dg}{dx}\Big|_{x=x_0}(x - x_0)$. Hence $\frac{dg}{dx}\Big|_{x=x_0}$ plays the role of k in the *simple* scaling property.

Multiple argument zeroes: In the case that the function $g(x)$ is equal to $g(x_0)$ for several values of x in the interval (a,b), the integral found by applying the advanced scaling rule to a small region about each zero and summing the contributions from each zero in the interval (a,b).

$$\int_a^b f(x)\ \delta\left[g(x) - g(x_0)\right]\ dx = \sum_{\substack{x_j \ni g(x_j) = g(x_0) \\ x_j \in (a,b)}} f(x_j)/\left|\frac{dg}{dx}\right|\Big|_{x=x_j}$$

As the integration variable x is incremented positively and the delta is even, the procedure above provides positive weight to the value of $f(x)$ at each root of $g(x) - g(x_o)$.

SAMPLE CALCULATION: Delta function of an argument with multiple zeroes:

Anchor Step: *Identify the set of values of the integration variable for which the argument of the delta function is zero*. Identify the subset of these values that lie in the range of the integration.

Consider $I = \int_{-\infty}^{\infty} f(x)\; \delta(x^2 - 2)\, dx$. The argument of the delta function has zeroes for $x = \pm\sqrt{2}$.

Restrict your attention to the subset of those values that lie in the integration range. In this case the two values, $x = \pm\sqrt{2}$, are in the range. The integral can be evaluated considering the small regions about $x = \pm\sqrt{2}$ and the contributions evaluated using the advanced scaling rule.

$$I = \int_{-\infty}^{\infty} f(x)\; \delta(x^2 - 2)\, dx = \int_{-\sqrt{2}-\varepsilon}^{-\sqrt{2}+\varepsilon} f(x)\; \delta(x^2 - 2)\, dx + \int_{\sqrt{2}-\varepsilon}^{\sqrt{2}+\varepsilon} f(x)\; \delta(x^2 - 2)\, dx$$

This form is chosen to emphasize the action of the delta function. It provides net integrated weight to the factor in the integrand that it multiplies values in infinitesimal neighborhoods of the zeroes of its argument. The delta function has value zero outside these infinitesimal regions, and so the behavior and value of *f*(*x*) outside these regions is of no consequence.

Advanced scaling $\Rightarrow \dfrac{dg(x)}{dx} = 2x \;\Rightarrow \left|\dfrac{dg(x)}{dx}\right| = 2\sqrt{2}$ for $x = \pm\sqrt{2}$.

$$I = \int_{-\sqrt{2}-\varepsilon}^{-\sqrt{2}+\varepsilon} f(x)\; \left|\frac{dg(x)}{dx}\right|^{-1} \delta(x - \left[-\sqrt{2}\right])\, dx + \int_{\sqrt{2}-\varepsilon}^{\sqrt{2}+\varepsilon} f(x)\; \left|\frac{dg(x)}{dx}\right|^{-1} \delta(x - \sqrt{2})\, dx$$

$$I = \left[\frac{f(-\sqrt{2})}{\left|2\sqrt{2}\right|} + \frac{f(\sqrt{2})}{\left|2\sqrt{2}\right|} \right] \quad \textbf{\textit{For any continuous f(x).}}$$

FOUNDATION: The anchor step is crucial. Complete the anchor step first. Prepare an explicit list of the values of the integration variable that lie in the range of integration for which the argument of the delta function is zero. Proceed only after this step is complete and documented.

RULES SUMMARY: Apply after preparing a list of argument zero locations.

Defining Property: The Dirac delta ***function*** $\delta(x - x_0)$ is defined by the values of its integral.

$$\int_a^b \delta(x-x_0)\;dx = \begin{cases} 1 \; if \;\; x_0 \in (a,b) \\ 0 \; if \;\; x_0 \notin [a,b] \end{cases} \qquad \text{and } \; \delta(x-x_0)=0 \;\; for \;\; x \neq x_0$$

Derivative Property: $\int_a^b f(x)\,\frac{d}{dx}\left[\delta(x-x_0)\right]\,dx = -\left.\frac{df(x)}{dx}\right|_{x=x_0} \quad if \;\; x_0 \in (a,b)$

Scaling Property: $\int_a^b f(x)\;\delta(k\,[x-x_0])\;dx = (1/|k|)\quad f(x_0) \qquad if \;\; x_0 \in (a,b)$

Advanced Scaling: $\int_a^b f(x)\;\delta\left[g(x)-g(x_0)\right]\,dx \quad = \quad f(x_0)/\left|\frac{dg}{dx}\right|\Big|_{x=x_0} \qquad if \;\; x_0 \in (a,b)$

Multiple zeroes: $\int_a^b f(x)\;\delta\left[g(x)-g(x_0)\right]\,dx = \sum_{\substack{x_j \ni g(x_j)-g(x_0)=0 \\ x_j \in (a,b)}} f(x_j)/\left|\frac{dg}{dx}\right|\Big|_{x=x_j}$

Derivative Prop II: $\int_a^b f(x)\;\frac{d}{dx}\left(\delta\left[g(x)-g(x_0)\right]\right)\,dx = \sum_{\substack{x_j \ni g(x_j)-g(x_0)=0 \\ x_j \in (a,b)}} \left(-\frac{df(x_j)}{dx}\right) * \left(\left|\frac{dg}{dx}\right|\Big|_{x=x_j}\right)^{-1}$

WARNING: The scaling aspects are the most problematic for those only recently introduced to Dirac deltas. Be sure to include the simple scaling factor $|k|^{-1}$ or the advanced scaling factor $|{}^{dg}/_{dx}|^{-1}$ evaluated at each zero of the delta's argument. Be attentive; make the absolute value explicit even when it is not needed.

Endpoint argument zeros: If the argument of the delta function vanishes for an endpoint value of *x* (= *a* or *b*), the contribution to the integral is *usually* one-half the value that would be computed for an interior point. This result can be based on the ${}^1D_n(x)$ representation of the delta function discussed below. A digression on endpoint behavior follows.

> ***Deltas on the boundary**: The zero of the argument can appear on the boundary in 3D cases. Consider the space to be stacked coordinates 'cubes'. If the zero is on a face on an included cube, expect a relative weight of one-half. If the boundary zero is on the edge of an included cube, expect a relative weight of one-fourth. If the boundary zero is at a corner of the integration volume, expect a weight of one-eighth. Consider the solid angle Ω_{inc} about the zero that is included in the integration range. The relative weight expected is* ${}^{\Omega_{inc}}/_{4\pi}$.
> *A related issue arises for coordinates that have ranges bounded by finite values. In spherical coordinates, possible examples are $\delta(\phi - 2\pi)$ and $\delta(r - 0)$. In such cases, the*

integration range ends ***at a limit of the coordinate range****. The relevant evaluation follows by evaluating the delta at an interior point near the end of the range and then taking the limit that the point approaches that end limit. The first case $\delta(\phi - 2\pi)$ approaches the endpoint from below. The zero is to be offset slightly into the interior of the integration range, and then the limit that the offset approaches zero* is to be taken.*

$$\delta(\phi - 2\pi) \to \underset{\eta\to 0^+}{Limit}\left[\delta\left(\phi - [2\pi - \eta]\right)\right].$$

This procedure supports full weight for integration endpoints that are also at the finite limits of the full-range of the coordinate. Similarly, in a case in which the endpoint is approached from above: $\delta(r-0) \to \underset{\eta\to 0^+}{Limit}\left[\delta\left(r - [0+\eta]\right)\right]$.

Be skeptical! Verify that the total integrated weight of the delta function matches expectation. See problems 1, 2, 3 and 6 at the end of this section.

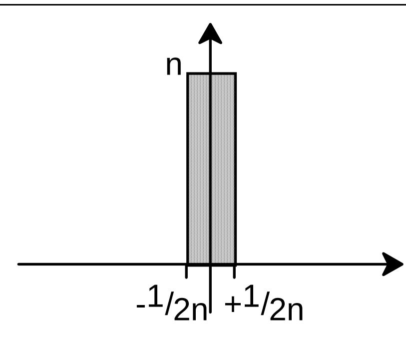
* See the example in the Tools of the Trade section. Look for equation: [DD.14]. The example suggests that any symmetry that is desired in the limit should be made explicit in the model prior to taking the limit.

The Dirac delta function $\delta(x - x_0)$ is zero everywhere except at the zero of its argument where it explodes huge positive. It manages this undefined act so as to have an area of 1 under its curve.

Bizarre as it is, the Dirac delta is not an ordinary function, but rather it has (many) representations as the ***limits of families of functions***. Such things are ***distributions* rather than *functions***. These families consist of well-defined function that exhibit *singular* behavior (blow up; have infinities) only in the limit in which they become the Dirac Delta.

Our first representation is the tall-rectangle **distribution**.

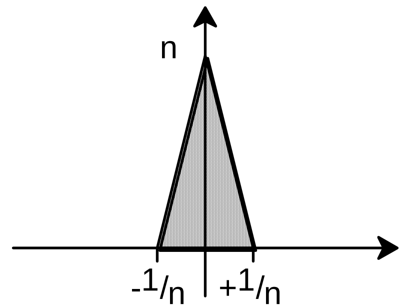
$$ {}^1D_n(x) = \begin{cases} 0 & \text{for } x < -\tfrac{1}{2n} \\ n & \text{for } -\tfrac{1}{2n} < x < +\tfrac{1}{2n} \\ 0 & \text{for } x > +\tfrac{1}{2n} \end{cases} $$



In the limit that $n \to \infty$, the action of the sequence of functions ${}^1D_n(x)$ approaches the behavior specified for the Dirac delta.

Our second representation is the tall-triangle distribution.

$$ {}^{2}D_n(x) = \begin{cases} 0 & \text{for } x < -1/n \\ n + n^2 x & \text{for } -1/n < x < 0 \\ n - n^2 x & \text{for } 0 < x < +1/n \\ 0 & \text{for } x > +1/n \end{cases} $$

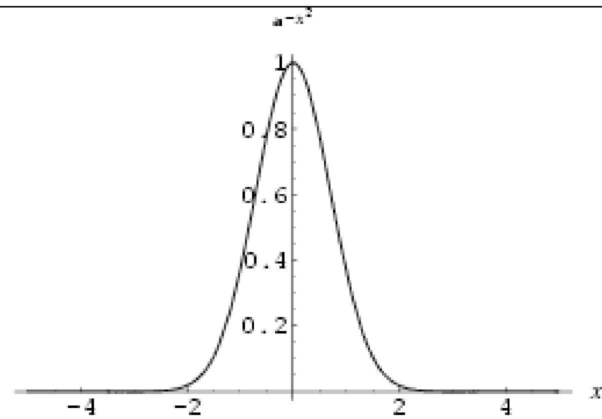


In the limit $n \to \infty$, the action of the sequence of functions ${}^{2}D_n(x)$ approaches the behavior specified for the Dirac delta.

Our third representation is the tall-Gaussian distribution.

$$ {}^{3}D_n(x) = \frac{n}{\sqrt{\pi}} \; e^{-(n\,x)^2} $$

See: mathworld.wolfram.com/



In the limit that $n \to \infty$, the sequence of functions ${}^{3}D_n(x)$ becomes tall and narrow, and its action approaches the behavior specified for the Dirac delta.

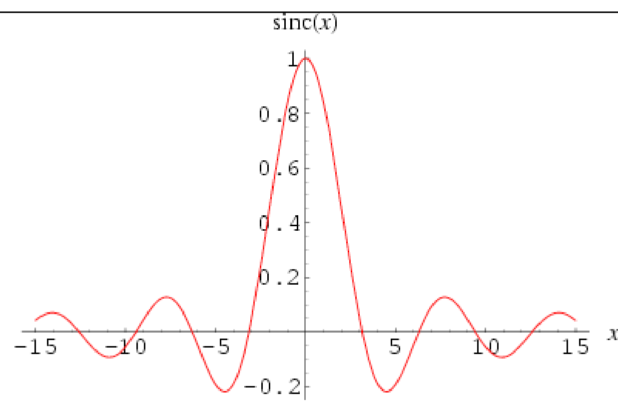
Our fourth representation is the tall-sinc distribution.

$$ {}^{4}D_n(x) = \left(\frac{n}{\pi}\right) \text{sinc}(nx) = \left(\frac{\sin(n\,x)}{\pi\;x}\right) $$

$$ \text{sinc}(x) = \sin(x)/x $$

See: mathworld.wolfram.com/

sinc: pronounced ‘sink’



In the limit that $n \to \infty$, the sequence of functions ${}^4D_n(x)$ becomes tall and narrow, and its action approaches the behavior specified for the Dirac delta.

Our fifth representation is the Fourier integral form.

$$ {}^5D_n(x-x_o) = \left(\frac{1}{2\pi}\right)\int_{-n}^{+n} e^{ik(x-x_o)}\,dk \;\to\; \left(\frac{1}{2\pi}\right)\int_{-\infty}^{+\infty} e^{ik(x-x_o)}\,dk = \delta(x-x_o) $$

Our sixth representation is as the derivative of the Heaviside function (the unit step function).

$$ \theta(x-x_0) = \begin{cases} 0 & for\;\; x < x_0 \\ 1 & for\;\; x > x_0 \end{cases} \quad \Rightarrow \quad \delta(x-x_0) = \frac{d\left[\theta(x-x_0)\right]}{dx} $$

The representation should be recast as a sequence! Perhaps using:

$$ \theta_n(x-x_0) = \begin{cases} 0 & for\;\; x < x_0 - \tfrac{1}{2n} \\ n(x-x_0)+\tfrac{1}{2} & for\; |x-x_0| < \tfrac{1}{2n} \\ 1 & for\;\; x > x_0 + \tfrac{1}{2n} \end{cases} $$

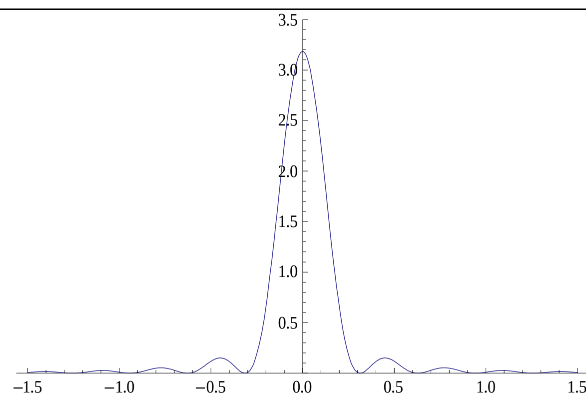
$$ \frac{d}{dx}\left(\left[\theta_n(x-x_0)\right]\right) = {}^1D_n(x-x_0) \quad \Rightarrow \quad \delta(x-x_0) = \underset{n\to\infty}{Limit}\left(\frac{d\left[\theta_n(x-x_0)\right]}{dx}\right) $$

Our seventh representation is the tall-sinc-squared distribution.

$$ {}^7D_n(x) = \left(\frac{n}{\pi}\right)\mathrm{sinc}^2(nx) = \left(\frac{\sin^2(n\,x)}{n\;\pi\;x^2}\right) $$

$$ \mathrm{sinc}(x) = \sin(x)/x $$

See: mathworld.wolfram.com/



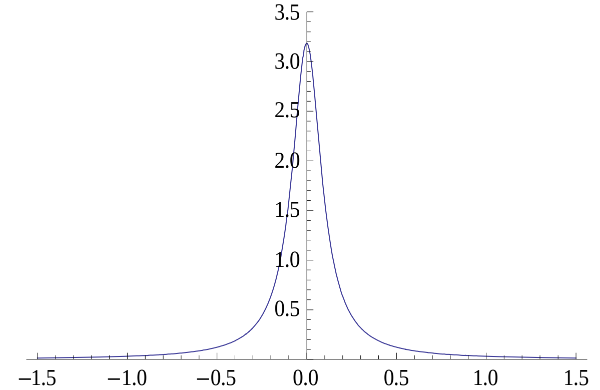
Plot[(Sin[10 x])^2/(10 Pi x^2),{x,-1.5,1.5}, PlotRange -> {0,3.5}]

In the limit that $n \to \infty$, the sequence of functions ${}^7D_n(x)$ becomes tall and narrow, and its action approaches the behavior specified for the Dirac delta.

Our eighth representation is the resonance distribution.

$${}^{8}D_n(x) = \left(\frac{n}{\pi \left[1 + n^2 x^2 \right]} \right)$$

See: mathworld.wolfram.com/



Plot[10/(Pi (1+ 10^2 x^2)),{x,-1.5,1.5}, PlotRange -> {0,3.5}]

In the limit that $n \to \infty$, the sequence of functions ${}^{8}D_n(x)$ becomes tall and narrow, and its action approaches the behavior specified for the Dirac delta.

The ${}^{1}D_n(x)$ sequence can be used to illustrate the action of such a sequence in the limit of large n on a function described by its Taylor's series expansion. It is assumed that x_0 lies between a and b.

$$\int_a^b f(x)\; {}^{1}D_n(x)\, dx \to \int_{x_0 - 1/2n}^{x_0 + 1/2n} f(x)\; n \;\; dx \approx n \int_{x_0 - 1/2n}^{x_0 + 1/2n} \left[f(x_o) + \left.\tfrac{df}{dx}\right|_{x_o} \left(x - x_o \right) + \tfrac{1}{2!} \left.\tfrac{d^2 f}{dx^2}\right|_{x_o} \left(x - x_o \right)^2 + \ldots \right] \;\; dx$$

$$= n \int_{-1/2n}^{+1/2n} \left[f(x_o) + \left.\tfrac{df}{dx}\right|_{x_o} \left(u \right) + \tfrac{1}{2!} \left.\tfrac{d^2 f}{dx^2}\right|_{x_o} \left(u \right)^2 + \ldots \right] \; du = n \left[f(x_o)\; u + \left.\tfrac{df}{dx}\right|_{x_o} \tfrac{1}{2!} u^2 + \left.\tfrac{d^2 f}{dx^2}\right|_{x_o} \tfrac{1}{3!} u^3 + \ldots \right]_{-1/2n}^{+1/2n}$$

$$= n \left[f(x_o)\; \tfrac{1}{n} + \left.\tfrac{df}{dx}\right|_{x_o} \left(\tfrac{1}{8n^2} - \tfrac{1}{8n^2} \right) + \left.\tfrac{d^2 f}{dx^2}\right|_{x_o} \tfrac{1}{3!} \tfrac{1}{4n^3} + \ldots \right] = f(x_o) + \left.\tfrac{df}{dx}\right|_{x_o} (0) + \left.\tfrac{d^2 f}{dx^2}\right|_{x_o} \tfrac{1}{3!} \tfrac{1}{4n^2} + \ldots \underset{n \to \infty}{\longrightarrow} f(x_o)$$

Note that the odd terms vanish identically using ${}^{1}D_n(x)$.

Exercise: For the sinc distribution ${}^{4}D_n(x)$, find the locations of its first zeros to the left and right of $x=0$ as a function of n. Use L'Hospital's rule to find the value of ${}^{4}D_n(x=0)$ as a function of n.

Exercise: Given that the sinc distribution ${}^{4}D_n(x)$, is a representation of the delta function, show that the sinc-squared sequence ${}^{7}D_n(x)$ is also a representation of the delta function. Use L'Hospital's rule as needed.

Delta functions in several dimensions:

Defining Property: The 2D Dirac delta ***function*** $\delta^2(\vec{r}-\vec{r}_0)$ is defined by the value that results after under integration over an area.

$$\int_{area\ A} \delta^2(\vec{r}-\vec{r}_0)\ dA = \begin{cases} 1\ if\ \ \vec{r}_0 \in A \\ 0\ if\ \ \vec{r}_0 \notin A \end{cases} \qquad \text{[DD.6]}$$

Defining Property: The 3D Dirac delta ***function*** $\delta^3(\vec{r}-\vec{r}_0)$ is defined by the value that results after under integration over a volume.

$$\int_{volume\ V} \delta^3(\vec{r}-\vec{r}_0)\ dV = \begin{cases} 1\ if\ \ \vec{r}_0 \in V \\ 0\ if\ \ \vec{r}_0 \notin V \end{cases} \qquad \text{[DD.7]}$$

> *Beware*: The *n* dimensional Dirac delta $\delta^n(\vec{r}-\vec{r}_0)$ is often represented as: $\delta(\vec{r}-\vec{r}_0)$. The dimensionality must be assumed to be the dimensionality of the argument of the delta. The integration weight for zeroes of the argument on the boundary of the integration region must be according to the rules for *endpoint* or boundary zeroes.

The two and three-dimensional Dirac delta function have straight forward representations in terms of the 1D deltas in Cartesian coordinates.

$$\delta^2([x,y],[x_0,y_0]) = \delta(x-x_0)\ \delta(y-y_0) \quad \text{and} \quad \delta^3(\vec{r}-\vec{r}_0) = \delta(x-x_0)\ \delta(y-y_0)\ \delta(z-z_0)$$

It is apparent from the definition: $\int_a^b \delta(x-x_0)\ dx = \begin{cases} 1\ if\ \ x_0 \in (a,b) \\ 0\ if\ \ x_0 \notin [a,b] \end{cases}$ that the delta function has the dimensions of (*length*)$^{-1}$, the inverse of the dimension of *x*.

A delta function of some other argument δ(*u*) has the dimensions of u^{-1}. For reasons similar to those dictating developing a gradient in which each term has the same dimension, it is convenient and appropriate in the case of a locally orthonormal coordinate system to associate the metric scale factors with each factor representing a multi-dimensional Dirac delta.

A specific line element might be:

$$d\vec{r} = 1 dr\ \ \hat{e}_r + r\, d\theta\ \ \hat{e}_\theta + r\sin\theta\, d\phi\ \ \hat{e}_\phi$$

The general form is:

$$d\vec{r} = h_1(q_1,q_2,q_3)\,dq_1\;\hat{e}_1 + h_2(q_1,q_2,q_3)\,dq_2\;\hat{e}_2 + h_3(q_1,q_2,q_3)\,dq_3\;\hat{e}_3$$

Given the line element use:

$$\delta^3(\vec{r}-\vec{r}_0) = \left\{\frac{\delta(q_1-q_{10})}{h_1(q_1,q_2,q_3)}\right\}\;\left\{\frac{\delta(q_2-q_{20})}{h_2(q_1,q_2,q_3)}\right\}\;\left\{\frac{\delta(q_3-q_{30})}{h_3(q_1,q_2,q_3)}\right\} \qquad \text{[DD.8]}$$

When a delta on one coordinate is needed, use the full contents of the corresponding pair of braces above. Example: A thin, radial line with uniform charge density linear λ can be represented by the *volume* charge density $\rho(\vec{r}) = \lambda\;\left\{\frac{\delta(\theta-\theta_0)}{r}\right\}\;\left\{\frac{\delta(\phi-\phi_0)}{r\,\sin\theta}\right\}$ in spherical coordinates.

The delta functions are useful for representing surface charge densities that coincide with constant coordinate surfaces and line charges that lie along coordinate orbits. If the line or surface charges do not fit the coordinate system nicely and naturally, parameterize the line or surface using standard techniques. A volume integral delta function approach would be difficult in such cases.

Examples:

Point charge q at (q_{10};q_{20},q_{30}): $\rho(\vec{r}) = q\;\left\{\frac{\delta(q_1-q_{10})}{h_1(q_1,q_2,q_3)}\right\}\left\{\frac{\delta(q_2-q_{20})}{h_2(q_1,q_2,q_3)}\right\}\left\{\frac{\delta(q_3-q_{30})}{h_3(q_1,q_2,q_3)}\right\}$

Line of charge along a q_2 orbit: $\rho(\vec{r}) = \lambda(q_2)\;\left\{\frac{\delta(q_1-q_{10})}{h_1(q_1,q_2,q_3)}\right\}\;\left\{\frac{\delta(q_3-q_{30})}{h_3(q_1,q_2,q_3)}\right\}$

Surface charge on a q_2 surface: $\rho(\vec{r}) = \sigma(q_1,q_3)\;\left\{\frac{\delta(q_2-a)}{h_2(q_1,q_2,q_3)}\right\}$

Exercise: Verify that a delta function has dimensions equal to the inverse of those of its argument. Base your reasoning on the defining property: $\int_{u_0-\varepsilon}^{u_0+\varepsilon}\delta(u-u_0)\,du = 1$.

Exercise: Verify that each factor above has dimension (*length*)$^{-1}$ given that each delta function has dimensions equal to the inverse of those of its argument.

In cylindrical coordinates, the dimensionally organized representation for the 3D Dirac delta is:

$$\delta^3(\vec{r}-\vec{r}_0) = \{\delta(r-r_0)\} \quad \left\{\frac{\delta(\phi-\phi_0)}{r}\right\} \quad \{\delta(z-z_0)\}$$

Exercise: Give the dimensionally organized representation for the 3D Dirac delta in spherical coordinates.

Green Functions and the Dirac Delta

Consider any linear driven differential equation $\hat{L}\left[x(t)\right] = f(t)$ where $\hat{L}$ represents the linear differential operator (derivatives with respect to *t*) and *f*(*t*) is the driving function. If one can solve the problem: $\hat{L}\left[G(t,t')\right] = \delta(t-t')$, then a formal solution to $\hat{L}\left[x(t)\right] = f(t)$ is:

$$x(t) = \int_{-\infty}^{\infty} G(t,t')\, f(t')\, dt' \text{ as}$$

$$\hat{L}\left\{\int_{-\infty}^{\infty} G(t,t')\, f(t')\, dt'\right\} = \int_{-\infty}^{\infty}\left\{\hat{L}[G(t,t')]\right\} f(t')\, dt' = \int_{-\infty}^{\infty} \delta(t-t')\, f(t')\, dt' = f(t).$$

Note the differential operator $\hat{L}$ operates on *t* and not on t'.

The general search for Green functions is to be postponed, but one rather advanced, important example is to be studied. The case of the Laplace equation in three dimensions takes the form:

$$\nabla^2 G(\vec{r},\vec{r}_s) = \delta^3\left(\vec{r}-\vec{r}_s\right)$$

The symbol $\vec{r}_s$ is a source position and $\vec{r}$ is the field position. The Laplacian is a second order differential operator that acts of the field position variables in $\vec{r}$. The Green function is to be deduced from facts previously established. The study of vector calculus has yielded the following information:

$$\vec{\nabla}\left(\frac{1}{\left|\vec{r}-\vec{r}_s\right|}\right) = -\frac{\vec{r}-\vec{r}_s}{\left|\vec{r}-\vec{r}_s\right|^3} \qquad \text{and} \qquad \oint_{\partial V} \vec{F}\cdot\hat{n}\, dA = \int_V \vec{\nabla}\cdot\vec{F}\, dV$$

where the surface integration is over the surface ∂V that encloses the integration volume for the volume integration V. Adding the vector calculus magic $\nabla^2 = \vec{\nabla}\cdot\vec{\nabla}$,

$$\oint_{\partial V}\left(\frac{\vec{r}-\vec{r}_s}{\left|\vec{r}-\vec{r}_s\right|^3}\right)\cdot\hat{n}\, dA = -\int_V \vec{\nabla}\cdot\vec{\nabla}\left(\frac{1}{\left|\vec{r}-\vec{r}_s\right|}\right) dV = -\int_V \nabla^2\left(\frac{1}{\left|\vec{r}-\vec{r}_s\right|}\right) dV \text{ or}$$

$$\int_V \nabla^2\left(\frac{1}{|\vec r-\vec r_s|}\right)dV=-\oint_{\partial V}\left(\frac{\vec r-\vec r_s}{|\vec r-\vec r_s|^3}\right)\cdot\hat n\,dA$$

The surface integral on the right was studied in gruesome detail in the vector calculus handout.

$\oint_{\partial V}\left(\frac{\vec r-\vec r_s}{|\vec r-\vec r_s|^3}\right)\cdot\hat n\,dA=\int_V\vec\nabla\cdot\left(\frac{\vec r-\vec r_s}{|\vec r-\vec r_s|^3}\right)dV$ and $\vec\nabla\cdot\left(\frac{\vec r-\vec r_s}{|\vec r-\vec r_s|^3}\right)\equiv 0$ *for* $\vec r\neq\vec r_s$. Although the divergence vanishes everywhere except perhaps at the point $\vec r=\vec r_s$, the surface integral gives a non-zero value if $\vec r_s\in V$ no matter how small the volume is chosen. A small spherical surface centered on $\vec r_s$ can be chosen in which case the evaluation is relatively simple (See the exercise below.).

$$-\oint_{\partial V}\left(\frac{\vec r-\vec r_s}{|\vec r-\vec r_s|^3}\right)\cdot\hat n\,dA=\int_V\nabla^2\left(\frac{1}{|\vec r-\vec r_s|}\right)dV=-4\pi\text{ if }\vec r_s\in V$$

But wait! The integrand $\nabla^2\left(\frac{1}{|\vec r-\vec r_s|}\right)$ vanishes everywhere except at the point $\vec r=\vec r_s$ and the integral over any volume that contains that point is – 4π. The conclusion is that

$\nabla^2\left(\frac{1}{|\vec r-\vec r_s|}\right)=-4\pi\,\delta^3(\vec r-\vec r_s)$ and hence that $G(\vec r,\vec r_s)=-\frac{1}{4\pi\,|\vec r-\vec r_s|}$.

Exercise: Transform to a set of spherical coordinated centered on $\vec r_s$ so that $\vec r$ represents $\vec r-\vec r_s$ for this exercise. Compute $\oint(\vec r/r^3)\cdot\hat n\;dA$ over a spherical surface of radius *R* centered on the origin directly as a surface integral; do not use Gauss's theorem. Does the result depend on *R*. Considering the *R* dependence of the result, describe the region responsible for the net value of the integral?

Application to electrostatics:

$$\vec\nabla\cdot\vec E=\rho_0/\varepsilon_o\qquad\vec E=-\vec\nabla\Phi\qquad\nabla^2\Phi=-\rho_0/\varepsilon_o\qquad\nabla^2\left(\frac{-1}{4\pi|\vec r-\vec r_s|}\right)=\delta(\vec r-\vec r_s)$$

A Green function for the Laplace equation is: $\frac{-1}{4\pi|\vec r-\vec r_s|}$ and $\nabla^2\Phi=-\rho_0/\varepsilon_o$ so,

$$\Phi(\vec r)=\int_V\left(\frac{-1}{4\pi|\vec r-\vec r_s|}\right)\left[-\rho_0(\vec r_s)/\varepsilon_o\right]d^3\vec r_s=\int_V\left(\frac{\rho_0(\vec r_s)}{4\pi\,\varepsilon_o|\vec r-\vec r_s|}\right)d^3\vec r_s\quad\text{[DD.9]}$$

The symbol $d^3\vec{r}_s$ indicates that the volume integration is over the source coordinates.

Similarly, in magnetostatics, $\vec{\nabla}\times\vec{B}=\mu_0\,\vec{J}$ and $\vec{\nabla}\times\vec{A}=\vec{B}$. Using the identity $\vec{\nabla}\times(\vec{\nabla}\times\ )=\vec{\nabla}(\vec{\nabla}\cdot\ )-\nabla^2$ and choosing $\vec{\nabla}\cdot\vec{A}=0$, the relations combine to give $\nabla^2\vec{A}=-\mu_0\,\vec{J}$ which following the development above means:

$$\vec{A}(\vec{r})=\int_V\left(\frac{\mu_0\,\vec{J}(\vec{r}_s)}{4\pi\,|\vec{r}-\vec{r}_s|}\right)d^3\vec{r}_s \qquad \text{[DD.10]}$$

To finish, one computes the negative gradient of the Φ to yield:

$$\vec{E}(\vec{r})=\int_V\left(\frac{\rho(\vec{r}_s)\,(\vec{r}-\vec{r}_s)}{4\pi\,\varepsilon_o\,|\vec{r}-\vec{r}_s|^3}\right)d^3\vec{r}_s \qquad \text{[DD.11]}$$

With slightly more effort, the curl of the vector potential $\vec{A}(\vec{r})$ yields:

$$\vec{B}(\vec{r})=\int_V\left(\frac{\mu_0\,\vec{J}(\vec{r}_s)\times(\vec{r}-\vec{r})}{4\pi\,|\vec{r}-\vec{r}_s|^3}\right)d^3\vec{r}_s \qquad \text{[DD.12]}$$

For currents that are confined to a filamentary path, the last equation becomes:

$$\vec{B}(\vec{r})=\int_V\left(\frac{\mu_0\,I\,d\vec{\ell}\times(\vec{r}-\vec{r})}{4\pi\,|\vec{r}-\vec{r}_s|^3}\right) \qquad \text{[DD.13]}$$

The expression is recognized as the Law of Biot and Savart.

Digression: A point about the theory of electromagnetism – Skip this discussion!

The divergence $\vec{\nabla}\cdot\vec{A}$ can be chosen to be zero because it has not direct physical interpretation. The curl $\vec{\nabla}\times\vec{A}$ is $\vec{B}$, but $\vec{\nabla}\cdot\vec{A}$ is just $\vec{\nabla}\cdot\vec{A}$. In fact, $\vec{\nabla}\cdot\vec{A}$ can be set to any value (function of position). One must not change $\vec{\nabla}\times\vec{A}$. The Helmholtz theorem provides the justification for this assertion. Again from vector calculus, this is equivalent to saying that the gradient of any scalar function can be added to $\vec{A}$ without changing the physics.

$$\vec{A}\,'=\vec{A}+\vec{\nabla}\psi \qquad \text{recall that } \vec{\nabla}\times\vec{\nabla}\psi\equiv 0$$

The process of adjusting $\vec{\nabla}\cdot\vec{A}$ is called *setting the gauge* and, although it is a frightening procedure, it is 100% analogous to adding an arbitrary constant to Φ to make the potential absolute. The gauge choice $\vec{\nabla}\cdot\vec{A}=0$ is called the Coulomb (or electrostatic) gauge, and it is a standard choice for static problems guaranteeing that both Φ and $\vec{A}$ satisfy a Poison's equation. For the case of electromagnetic radiation, the Lorenz[1] (or radiation) gauge $\vec{\nabla}\cdot\vec{A} = -\mu_0\varepsilon_0\,\frac{\partial\Phi}{\partial t}$ is chosen so that both Φ and $\vec{A}$ satisfy an equation of the same form, a driven wave equation.

$$\vec{\nabla}\cdot\vec{A} = -\mu_0\varepsilon_0\,\frac{\partial\Phi}{\partial t} \quad \Rightarrow \left[\nabla^2 - \mu_0\varepsilon_0\,\frac{\partial^2}{\partial t^2}\right]\Phi = -\rho/\varepsilon_0 \quad \text{and} \quad \left[\nabla^2 - \mu_0\varepsilon_0\,\frac{\partial^2}{\partial t^2}\right]\vec{A} = -\mu_0\,\vec{J}$$

**** 2D Delta (Lea): $\nabla^2[\ln(\rho/a)] = 2\pi\,\delta^{[2]}(\vec{\rho})$ *where* $\vec{\rho} = (x-x_0)\,\hat{i} + (y-y_0)\,\hat{j}$

Complete Sets of Functions and the Dirac Delta

Suppose that a set of functions $\{\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x), \ldots\}$ is a complete orthogonal set of basis functions spanning a vector (function) space with the inner product:

$$\langle f(x)|g(x)\rangle = \int f^*(x)\; g(x)\,w(x)\,dx$$

An arbitrary function cab be represented as: $f(x) = \sum_n a_n\,\varphi_n(x)$. The Dirac delta is not a well-behaved function, but if it can be expanded formally as:

$$\delta(x-x_0) = \sum_n b_n\,\varphi_n(x)\,.$$

Project out the coefficient b_m by pre-multiplication by $\varphi_m^*(x)$ and applying the inner product procedure, integration of the product over the range of x yields:

$$\int \varphi_m^*(x)\;\delta(x-x_0)\,w(x)\,dx = \varphi_m^*(x_0) = \sum_n b_n \int \varphi_m^*(x)\,\varphi_n(x)\,w(x)\,dx = b_m \int \varphi_m^*(x)\,\varphi_m(x)\,w(x)\,dx$$

$$b_m = \frac{\varphi_m^*(x_0)\,w(x_0)}{\int \varphi_m^*(x')\,\varphi_m(x')\,dx'}$$

Substituting for b_m,

[1] The Lorenz gauge is often identified incorrectly as the Lorentz gauge.

$$\delta(x-x_0)=\sum_m\left(\frac{\varphi_m^*(x_0)\,w(x_0)}{\int\varphi_m^*(x')\,\varphi_m(x')\,dx'}\right)\varphi_m(x)$$

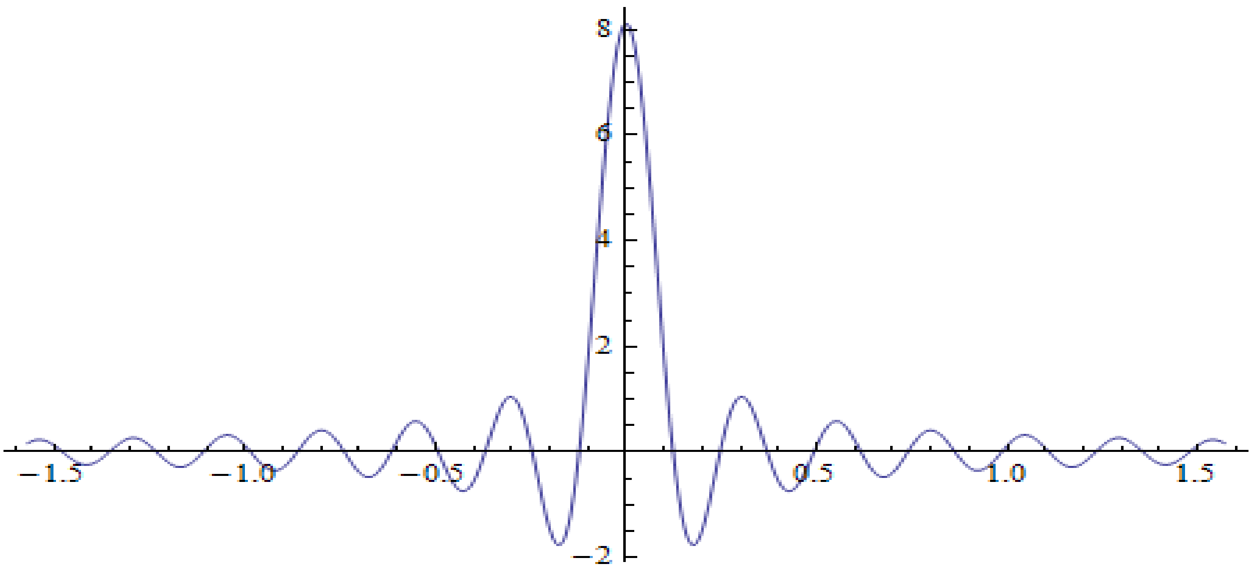
WARNING: The rapid extreme rapid variation of the delta means that its spectrum is infinitely broad. No finite number of terms is adequate. The relation above is useful formally where a sum over all the eigenfunctions can be imagined.

Fourier series example: The delta will be centered on zero so that, as an even function, it will have only constant and cosine character. For a period of π, it follows that $c_o = {}^1/_{2\pi}$ and $a_n = {}^1/_{\pi}$ for all n.

** One must include the sine terms to represent $\delta(t - t_o)$ for $t_o \neq 0$.

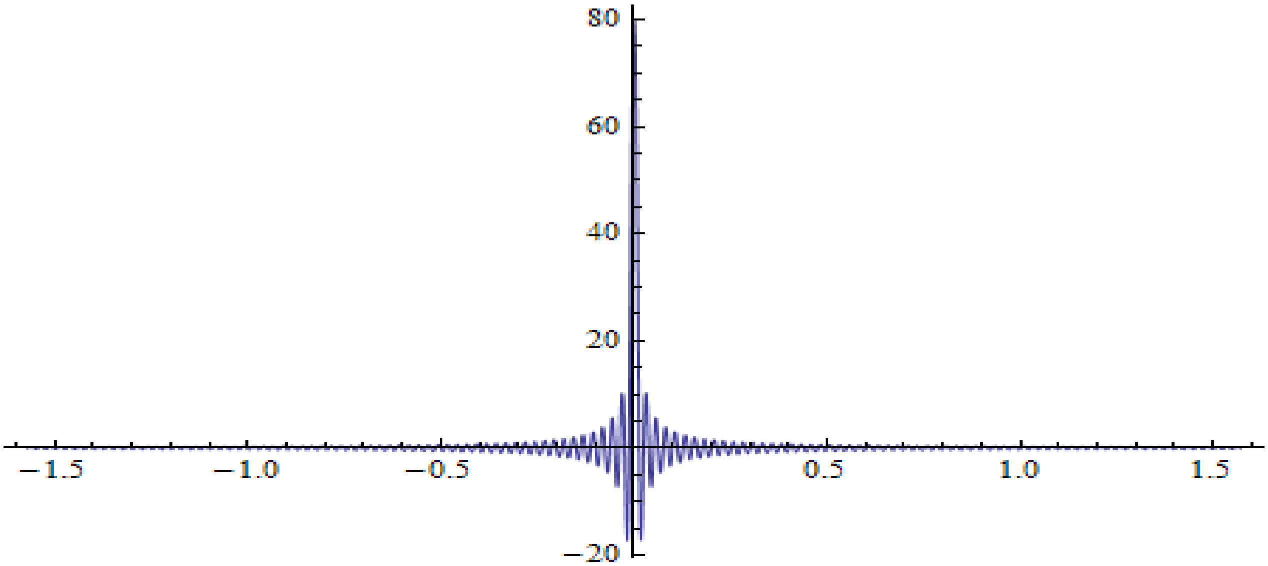
25 + 1 Term Expansion:
Fourier Trig

```
f[t_] = ((1/2) + Sum[Cos[n t],{n,1,25}])/Pi;
Plot[f[t],{t,-Pi/2, Pi/2}, PlotRange → All]
```



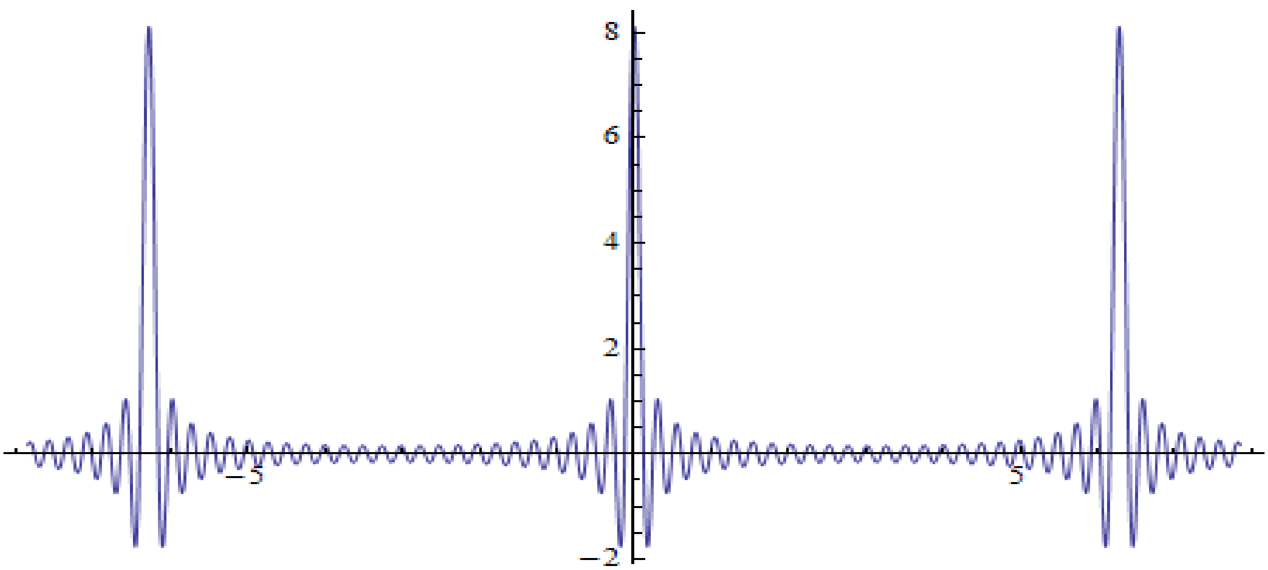
250 + 1 Term Expansion:
≈ 10x higher; narrower

```
f[t_] = ((1/2) + Sum[Cos[n t],{n,1,250}])/Pi;
Plot[f[t],{t,-Pi/2, Pi/2}, PlotRange → All]
```




```
f[t_] = ((1/2) + Sum[Cos[n t],{n,1,25}])/Pi;
Plot[f[t],{t,-2.5  Pi, 2.5  Pi}, PlotRange → All]
```

Fourier series represent periodic function so the delta repeats period after period. This can lead to unpleasant outcomes if the recurrence behavior is not appropriate for the application.



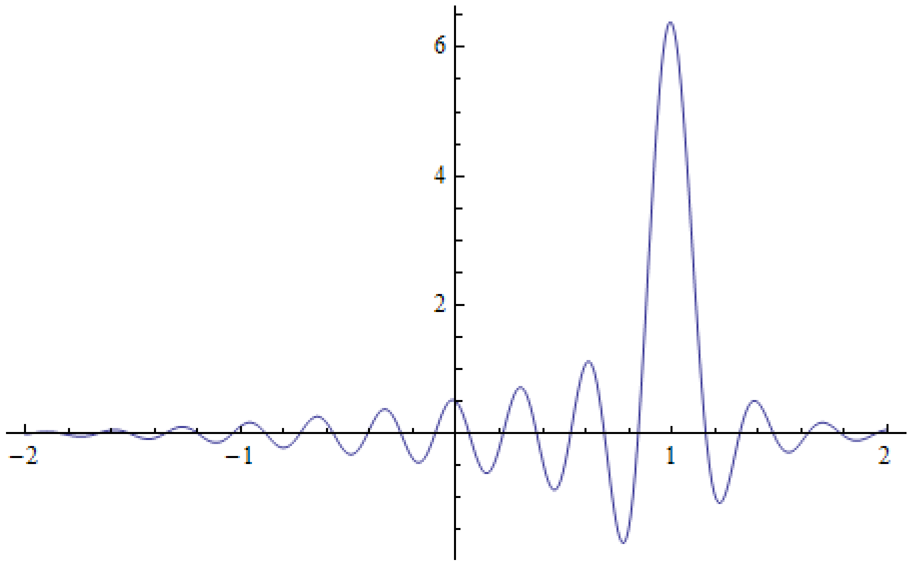
Consider the case of the Harmonic oscillator wave functions.
Using Mathematica, an expression for the delta function is generated by summing over the first 200 harmonic oscillator wavefunctions with $x_o = 1$.

Hermite (QSHO) example: $\delta(x-1) \approx \sum_{m=0}^{200} \varphi_m^*(1\,)\,\varphi_m(x)$ where $\varphi_m(x)$ is the simple harmonic oscillator wave function for the state with energy $(m + ½)\,\hbar\,\omega_o$. In principle,

$$\delta(x-1) \approx \sum_{m=0}^{\infty} \varphi_m^*(1\,)\,\varphi_m(x)\,.$$

$$\varphi_n(x) = \frac{1}{\sqrt{2^n\,n!\sqrt{\pi}}}\,e^{-½x^2}\;H_n(x)$$

```
td[n_,x_] =  Exp[-x^2] HermiteH[n,1] HermiteH[n,x]/(2^n n!  Sqrt[Pi])
dd[x_] =Sum[td[n,x],{n,0,200}];
Plot[dd[x],{x,-2,2}, PlotRange -> All]
```



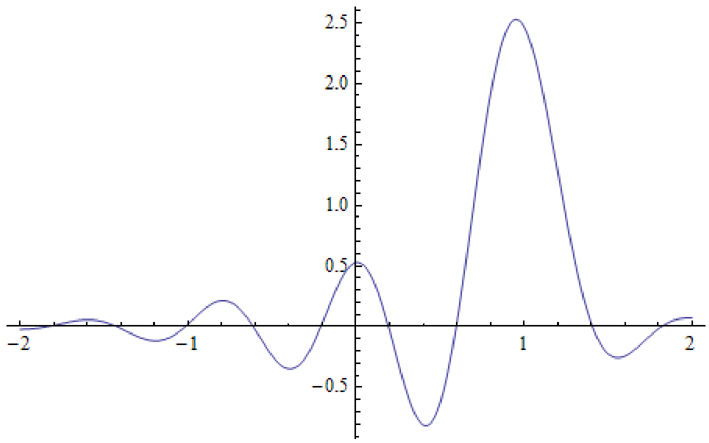
NIntegrate[dd[x],{x,-1,2}]   = 1.00338

The function is small and oscillatory except in a small neighborhood of 1. The net area under the curve is close to 1. It's approaching a $\delta(x - 1)$.

Comparison with the 31 term attempt.

dd[x_] =Sum[td[n,x],{n,0,30}];

Plot[dd[x],{x,-2,2}, PlotRange -> All]

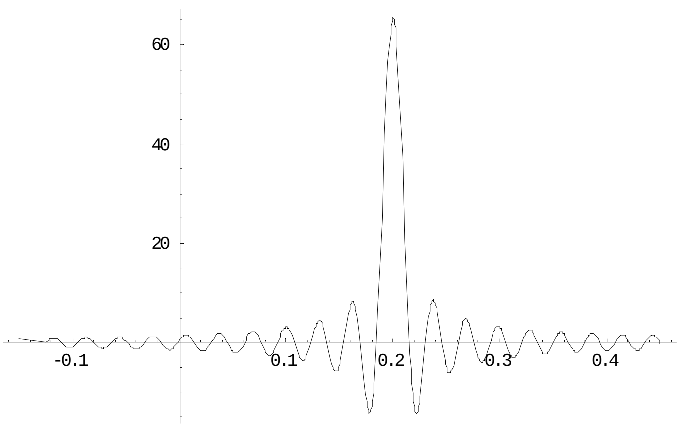


NIntegrate[dd[x],{x,-2,2}] ⇒ 0.997079

Legendre polynomial example: 201 terms

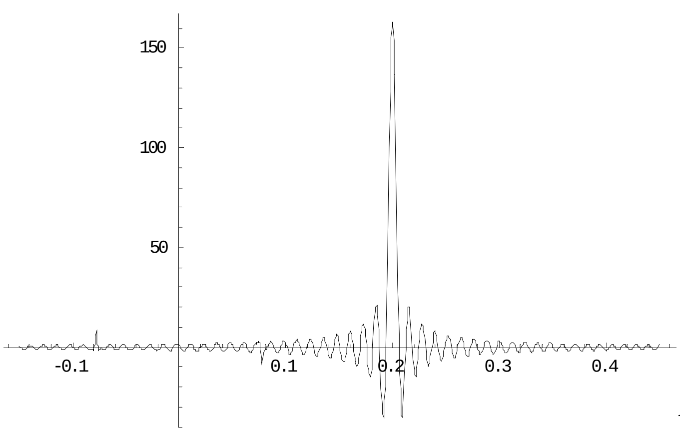
deltaLegendre[x_, xL_] := Sum[(LegendreP[m,x] LegendreP[m,xL]*(2 m + 1)), {m, 0, 200}]/2;

Plot[deltaLegendre[x,xL], {x, -.15, .45}, PlotRange → All]



Integrate[delta2Legendre[x], {x, 0.18, 0.22}] = 0.948105 200 terms; narrow integration range

Legendre polynomial example: 501 terms



taller and narrower!

Recall that the Dirac delta function **does not** meet the requirements to have a faithful expansion in terms of our complete set so such expansions must be considered as suspect. It is important to realize that ${}^{5}D_n(x\text{-}x_0)$ is of the complete set expansion form, the Fourier form. The sum is over a continuous label and so is an integral.

$$ {}^{5}D_n(x-x_o) = \left(\frac{1}{2\pi}\right)\int_{-n}^{+n} e^{ik(x-x_o)}\, dk \to \int_{-\infty}^{+\infty} e^{ik(x-x_o)}\, dk = \int_{-\infty}^{+\infty} \left(e^{ikx_o}\right)^* e^{ikx}\, dk = 2\pi\ \delta(x-x_o) $$

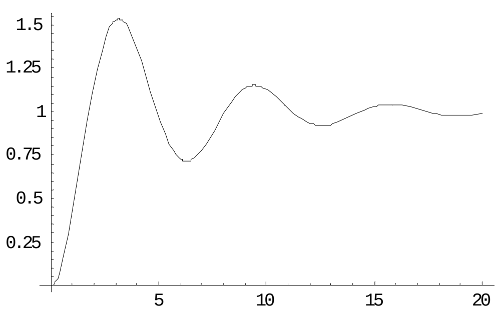
$$ \delta(x-x_0) = \sum_m \varphi_m^*(x_0)\varphi_m(x) \to \delta(x-x_o) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(e^{ikx_o}\right)^* e^{ikx}\, dk $$

Tools of the Trade

Develop the Green function for the damped, driven oscillator by using the difference of the responses to a positive step followed by an equal magnitude negative step.
See is a second approach follows by taking a formal derivative of the step function response.

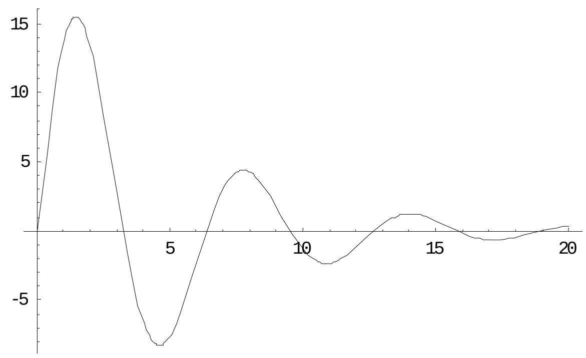
```
w0 = 1; beta = 0.2; w1 = Sqrt[w0^2 - beta^2];dt = .2; amp = 20/dt;
            xr2[t_]:=amp * (1 +  Exp[- beta t](- Cos[t]/w0^2 - beta * Sin[t]/(w1 w0^2)))
```

```
Plot[xr[t],{t,0,20}]
```



```
xr2[t_]:=amp * (1 +  Exp[- beta t](- Cos[t]/w0^2 - beta * Sin[t]/(w1 w0^2))
     -UnitStep[t-dt]*( 1 +  Exp[- beta (t-dt)](- Cos[t-dt]/w0^2 - beta * Sin[t-dt]/(w1 w0^2))))
```

```
Plot[xr2[t],{t,0,20}, PlotRange → All]
```



Make *dt* shorter and amp greater until impulse response remains 'fixed as *dt* is decreased.

Representing singular current distributions: (response to Midn Follador's question)

A line current *I* running along the *z* axis (cylindrical coordinates) presents several issues. It involves a delta function with at argument zero for *r* = 0. That is: as the end of the range of the coordinate variable. The second is that for many purposes, the problem must display cylindrical symmetry. Ignoring symmetry, one handles a delta at the end of a coordinate run by displacing slightly from the end and then taking the limit that the displacement approaches zero.

$$J_z(r,\phi) = I\,\delta(r-\varepsilon)\,\frac{\delta(\phi-\phi_0)}{r}$$

To verify the form, the current density is integrated over the plane to check the total current.

$$\int J\,dA = \int_0^\infty \int_0^{2\pi} I\,\delta(r-\varepsilon)\,\frac{\delta(\phi-\phi_0)}{r}\; r\,d\phi\,dr = I$$

It works, but it is not cylindrically symmetric. For that, $J_z(r,\phi) = I\,\dfrac{\delta(r-\varepsilon)}{2\pi\,\varepsilon}$ [DD.14]. This represent a thin-walled cylindrical pipe of radius ε carrying a current *I*. The limit $\varepsilon \to 0$ is to be taken as a final step to reach the thin line of current limit.

Ampere's Law is $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ so $[\vec{\nabla}\times\vec{B}]_z = \mu_0 J_z$. In cylindrical coordinates,

$$\vec{\nabla}\times\vec{F} = \left[\left(\frac{1}{r}\right)\frac{\partial F_z}{\partial\phi} - \frac{\partial F_\phi}{\partial z}\right]\hat{r} + \left[\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right]\hat{\phi} + \left(\frac{1}{r}\right)\left[\frac{\partial\left(r\,F_\phi\right)}{\partial r} - \frac{\partial F_r}{\partial\phi}\right]\hat{k}$$

so $\left(\dfrac{1}{r}\right)\left[\dfrac{\partial\left(r\,B_\phi\right)}{\partial r} - \dfrac{\partial B_r}{\partial\phi}\right] = \mu_0 J_z(r,\phi) = \mu_0\,I\,\dfrac{\delta(r-\varepsilon)}{2\pi\,r}$. There can be no ϕ dependence in a cylindrically symmetric problem so the law reduces to:

$$\left(\frac{1}{r}\right)\frac{\partial\left(r\,B_\phi\right)}{\partial r}=\mu_0 J_z(r)=\mu_0\, I\;\frac{\delta(r-\varepsilon)}{2\pi\, r}\;\text{ or }\;\frac{d\left(r\,B_\phi\right)}{dr}=(2\pi)^{-1}\;\mu_0\, I\;\delta(r-\varepsilon)$$

The derivative becomes a total vice a partial because the functions only depend on *r*. In this form, the current density at *r* = 0 is 0. This value is non-singular so the field at *r* = 0 must be able to choose a direction if it is to be non-zero. As the is no basis for a choice, $B_\phi(0) = 0$.

$$r\,B_\phi(r)=\left[r\,B_\phi(r)\right]\Big|_{r=0}+\int_0^r\frac{d\left(r'\,B_\phi(r')\right)}{dr'}\,dr'=\left[r\,B_\phi(r)\right]\Big|_{r=0}+(2\pi)^{-1}\;\mu_0\, I\;\int_0^r\delta(r'-\varepsilon)\,dr'$$

Recognizing that $r\,B_\phi = 0$ for $r = 0$,

$$r\,B_\phi(r)=(2\pi)^{-1}\;\mu_0\, I\;\int_0^r\delta(r'-\varepsilon)\,dr'=\begin{cases}0 & (r<\varepsilon)\\ (2\pi)^{-1}\;\mu_0\, I & (r>\varepsilon)\end{cases}$$

We conclude that:

$$\vec{B}(r)=\begin{cases}0 & (r<\varepsilon)\\ \left(\mu_0\, I/2\pi\, r\right)\;\hat{\phi} & (r>\varepsilon)\end{cases}$$

Finally, the limit that $\varepsilon \to 0$ can be taken. The final result for a super thin wire carrying a current *I* along the *z* axis is: $\vec{B}(r)=\left(\mu_0\, I/2\pi\, r\right)\;\hat{\phi}$.

Sample Calculations:

SC1.) $\int_3^8 \sin(\pi x)\,\delta(x-7/2)\,dx=\sin(7\pi/2)=-1$. The zero is at $x = 7/2$ which is in the integration range.

The argument of the delta function is *simple* so a direct application of the defining property evaluates the rest of the integrand for $x = 7/2$.

Mathematica: Integrate[Sin[Pi x] DiracDelta[x - 7/2],{x,3,8}] = -1

SC2.) $\int_0^{2\pi}\cos(\theta)\,\delta(3[\theta-\pi])\,d\theta=\frac{\cos(\pi)}{|3|}=-\frac{1}{3}$. The zero occurs for $\theta = \pi$ which is inside the integration range. The argument of the delta has a constant scaling factor 3. The result is the inverse of the absolute value of the scaling factor times the remainder of the integrand evaluated for $\theta = \pi$.

Integrate[Cos[theta] DiracDelta[3(theta - Pi)],{theta,0,2 Pi}] = -1/3

SC3.) $2\sqrt{2}\int_0^2 e^{x^2/2}\;\; \delta(x^2-2)\,dx$. The argument has zeros for $x = \pm\sqrt{2}$. Only the positive value lies in range. The argument of the delta is a function $g(x) = x^2$ so the advanced scaling rule directs that the inverse of $|{}^{dg}/_{dx}| = |2x|$ evaluated at the zero, $x = +\sqrt{2}$, multiplies the value of the remainder of the integrand evaluated at $x = +\sqrt{2}$.

$$2\sqrt{2}\int_0^2 e^{x^2/2}\;\; \delta(x^2-2)\,dx = 2\sqrt{2}\left(\frac{1}{|2\sqrt{2}|}\right)e^{(\sqrt{2})^2/2} = e$$

2 Sqrt[2] Integrate[Exp[x^2/2] DiracDelta[x^2 - 2],{x,0,2}] = e

SC4.) $\int_0^4 \left(4x^{-2}\right)\frac{d\left[\delta(x-2)\right]}{dx}\,dx = -\left.\frac{d(4x^{-2})}{dx}\right|_{x=2} = -(-8x^{-3})\big|_{x=2} = 1$. The zero at $x = 2$ is in the range 0 to 4. The delta has a simple argument so the result is the negative of the derivative of the remainder of the integrand evaluated at $x = 2$.

Integrate[4 x^(-2)D[DiracDelta[x- 2],x],{x,0,4}] = 1

SC5.) $\int_{0.25}^{2.25}\cos(x)\;\; \delta(\sin(\pi x))\,dx$. The argument of the delta has zeros for all integer values of x. The integers 1 and 2 are in the range 0.25 to 2.25. The argument is a function, and advanced scaling directs that the inverse of $|{}^{dg}/_{dx}| = |\pi\cos(\pi x)|$ evaluated at each zeros multiplies the value of the remainder of the integrand evaluated at the corresponding zero. The process is simplified by noting that $|\pi\cos(\pi x)| = +\pi$ whenever $\sin(\pi x) = 0$.

$$\int_{0.25}^{2.25}\cos(x)\;\; \delta(\sin(\pi x))\,dx = \frac{\cos(1)}{|\pi\cos(\pi)|} + \frac{\cos(2)}{|\pi\cos(2\pi)|} = \frac{\cos(1)+\cos(2)}{\pi} \approx 0.03952$$

Integrate[Cos[x] DiracDelta[Sin[Pi x]],{x,.25,2.25}] = $\frac{\text{Cos[1]} + \text{Cos[2]}}{\pi}$ = 0.0395199…

SC6.) $\int_0^5 \cos(\pi x)\;\delta(x-1)\,\delta(x-4)\,dx = 0$. One or the other of the delta function is always zero so the result must be zero. For example when the leftmost delta has a zero of its argument, the rightmost delta has a value of zero. That rightmost delta is a factor in the remainder of the integrand so the net value is zero. Construct an argument supporting this interpretation based in the ${}^1D_n(x)$ representation of the delta function. *The use of a product of deltas should not be necessary and should be avoided.*

Integrate[Cos[Pi x] DiracDelta[x-1] DiracDelta[x-4],{x,0,5}]

Mathematica is a little unhappy!

SC7. $\int_{-4}^{4}\left(x^3\right)\ \tfrac{d}{dx}\left[\delta(x^2-4)\right]dx=\int_{-4}^{4}\delta(x^2-4)\left(-\tfrac{d}{dx}\left[x^3\right]\right)\ dx$. There are times when chaining together too many rules is confusing. Integrate the original problem by parts. The delta vanishes at the endpoints. Now apply the rules to: $\int_{-4}^{4}\delta(x^2-4)\left(-\tfrac{d}{dx}\left[x^3\right]\right)\ dx=\int_{-4}^{4}\delta(x^2-4)\left(-3x^2\right)\ dx$.

Using advanced scaling, $\int_{-4}^{4}\delta(x^2-4)\left(-3x^2\right)\ dx=\frac{1}{|2x|}\left(-3x^2\right)\Big|_{x=-2}+\frac{1}{|2x|}\left(-3x^2\right)\Big|_{x=+2}=-6$.

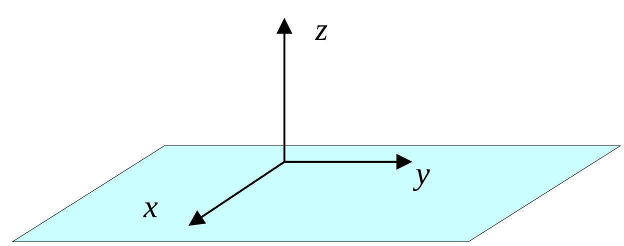
If you did not locate the zeroes and check to ensure that they were in the integration range, do so now.

Integrate[x^3 D[DiracDelta[x^2- 4],x],{x,-4,4}] = -6

WARNING: The scaling aspects are the most problematic for those only recently introduced to Dirac deltas. Be sure to include the simple scaling factor $|k|^{-1}$ or the advanced scaling factor $|{}^{dg}/_{dx}|^{-1}$ evaluated at each zero of the delta's argument. Be attentive; make the absolute value explicit even when it is not needed.

Sample Applications

SA1.) A portion of an infinite plane with surface charge density σ_0 is shown at right. Find the electric field $\vec{\boldsymbol{E}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ due to this plane using Gauss' Law in differential form. Briefly explain what you are doing, and remember to specify $\vec{\boldsymbol{E}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ for $z < 0$ and $z > 0$.



$$E_z(a)=E_z(-a)+\int_{-a}^{a}\frac{\partial E_z}{\partial z}\,dz=\int_{-a}^{a}\frac{\sigma_0}{\varepsilon_0}\,\delta(z-0)\,dz=E_z(-a)+\frac{\sigma_0}{\varepsilon_0}$$

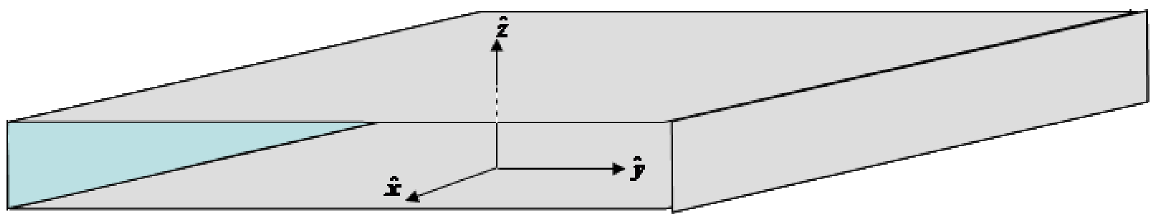
Combining this with the symmetry result, $E_z(a) = -E_z(-a)$ where a is an arbitrary and positive.

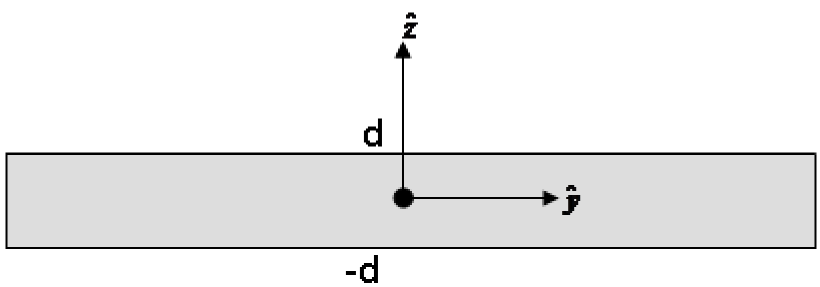
$$E_z(a) = -E_z(a) + \frac{\sigma_0}{\varepsilon_0} \text{ so } E_z(z) = \begin{cases} +\sigma_0/2\varepsilon_0 & \text{for } z > 0 \\ -\sigma_0/2\varepsilon_0 & \text{for } z < 0 \end{cases}$$

SA2.) Two views of a portion of an infinite charged slab are shown below: it extends infinitely far in the x and y directions, but is limited to $-\boldsymbol{d} \le \boldsymbol{z} \le \boldsymbol{d}$ in the z direction.

The slab's charge density is uniform with value ρ_0. Using Gauss' Law, find the E field inside and outside the slab.

.





In Cartesian coordinates with z dependence only, $\vec{\nabla} \cdot \vec{E} = \rho / \varepsilon_0 \rightarrow \frac{\partial E_z}{\partial z} = \rho(z)/\varepsilon_0$.

For $-d < z < d$, $\frac{\partial E_z}{\partial z} = \rho_0/\varepsilon_0 \Rightarrow E_z(z) = \rho_0 z/\varepsilon_0 + E_0$. By symmetry, $E_z(0) = 0$. The field cannot choose between $+z$ and $-z$ at $z = 0$ because the charge distribution is symmetric about the plane $z = 0$ so E_o = 0.

$$E_z(z) = \rho_0 z/\varepsilon_0 \quad \text{for } -d < z < d$$

For $|z| > d$, the charge density is zero so $\frac{\partial E_z}{\partial z} = \rho(z)/\varepsilon_0 = 0$ or E_z is constant in those regions.

Matching at the boundaries, the planes $z = -d$ and $z = d$, it follows that:

$$E_z(z) = \begin{cases} +\rho_o d/\varepsilon_0 & \text{for } z > d \\ +\rho_o z/\varepsilon_0 & \text{for } -d < z < d \\ -\rho_o d/\varepsilon_0 & \text{for } z < -d \end{cases}$$

Only a z component is *caused* so that is all there is *due to the specified charge distribution*.

Problems

1.) Integrate the cylindrical delta $\delta^3(\vec{r} - \vec{r}_0) = \{\delta(r - r_0)\} \left\{\frac{\delta(\phi - \phi_0)}{r}\right\} \{\delta(z - z_0)\}$ over all space to verify that its integral is one.

2.) Integrate the spherical delta $\delta^3(\vec{r} - \vec{r}_0)$ over all space to verify its integral is one.

3.) I claim that a thin spherical shell of uniform surface charge density σ can be represented by the *volume* charge density $\rho(\vec{r}) = \sigma \;\; \delta(r - R)$. Integrate this charge density over all space to find the net charge. Integrate the charge density over the volumes $r > (R+\varepsilon)$ and $r < (R-\varepsilon)$. Does the charge density represent a thin spherical shell of radius R with uniform surface charge density σ?

4.) The Fourier integral form is closely related to the sinc form.

Compute ${}^5D_n(x - x_o) = \left(\frac{1}{2\pi}\right)\int_{-n}^{+n} e^{ik(x-x_o)}\, dk$. Be cautious. The integral is with respect to k. The factor $(x - x_o)$ is a constant during the integration. Describe the relation between representations four and five.

5.) Work with ${}^2D_n(x)$, the triangular representation. Carefully determine the slope of ${}^2D_n(x)$ between n^{-1} and 0. Find a graphical representation for $\frac{d\left({}^2D_n(x - x_o)\right)}{dx}$. Express the result in terms of ${}^1D_n(x - x_a)$ for a few values of x_a. Replace each ${}^1D_n(x - x_a)$ by $\delta(x - x_a)$. Compute the integral of $\int_a^b f(x)\, \frac{d\left({}^2D_n(x - x_0)\right)}{dx}\, dx$ using the expression for $\frac{d\left({}^2D_n(x - x_0)\right)}{dx}$ in terms of a factor times

$\delta(x-x_a)$'s where the x_a are points slightly displaced from x_o. Complete the integration after replacing $\frac{d\left({}^2D_n(x-x_0)\right)}{dx}$ with the pair of delta functions Compare with

$$\int_a^b f(x)\,\frac{d\left[\delta(x-x_0)\right]}{dx}\,dx = \begin{cases} -\left.\frac{df}{dx}\right|_{x_0} & if \;\; x_0 \in (a,b) \\ 0 \quad if & x_0 \notin [a,b] \end{cases}$$

in the limit n large. (Review the definition of the derivative of $f(x)$ at x_o.)

Hint: $\frac{df}{dx} = \operatorname{Lim}_{\varepsilon\to 0} \frac{f(x+½\,\varepsilon) - f(x-½\,\varepsilon)}{\varepsilon}$; Replace ε by n^{-1}.

6.) I claim that a thin line of uniform charge density linear λ can be represented by the *volume* charge density $\rho(\vec{r}) = \lambda \left\{\frac{\delta(\theta-\theta_0)}{r}\right\} \left\{\frac{\delta(\phi-\phi_0)}{r\,\sin\theta}\right\}$ in spherical coordinates. Describe the line of charge. For example, let θ and ϕ be $^{\pi}/_{2}$. Integrate this charge density over volume of a sphere of radius R centered on the origin. Does the resulting net charge agree with expectations?

7.) In the study of electrostatics, the polarization charge density associated with a polarized dielectric is often represented by two contributions: a volume distributed charge density $\rho_{pol} = -\vec{\nabla}\cdot\vec{P}$ and, on the surface, a surface charge density $\sigma_{pol} = +\vec{P}\cdot\hat{n}$ where the outward directed normal is chosen at the surface and $\vec{P}$ is the polarization density of the dielectric. Consider a polarization density $\vec{P}$ that is uniform in the bulk of a cube of dielectric of side L centered on the origin with its faces parallel to the Cartesian coordinate planes. The polarization density is in the x direction and has magnitude P_0 in the inner regions of the block, and it falls linearly to zero in a thickness $^{L}/_{n}$ of the faces at $x = \pm {}^{L}\!/_{2}$. Prepare equations to represent $\vec{P}$ everywhere in the cube. Compute $\rho_{pol} = -\vec{\nabla}\cdot\vec{P}$ throughout the cube. What is the total charge in the slab of thickness $^{L}/_{n}$ at $x = +{}^{L}\!/_{2}$? …. at $x = -{}^{L}\!/_{2}$? In the limit n gets very large, use delta functions to represent ρ_{pol} where appropriate. Compare the result with $\sigma_{pol} = +\vec{P}\cdot\hat{n}$. The conclusion is that $\sigma_{pol} = +\vec{P}\cdot\hat{n}$ corresponds to the term in $\rho_{pol} = -\vec{\nabla}\cdot\vec{P}$ that could result from the derivative of a (Heaviside) step-function dependence at the surface. By adding $\sigma_{pol} = +\vec{P}\cdot\hat{n}$, you are relieved of the responsibility to compute the derivative of a discontinuous function. It is an alternative representation of the singular behavior of $\rho_{pol} = -\vec{\nabla}\cdot\vec{P}$ at a discontinuity.

8.) **Proving the Scaling Property**: Adopt the ${}^1D_n(x - x_o)$ representation and plot ${}^1D_n(k[x - x_o])$ where k is a constant. Find the net area under the curve for ${}^1D_n(k[x - x_o])$. Begin by plotting ${}^1D_n(x - x_o)$ and ${}^1D_n(k[x - x_o])$ for n = 9 and k = 3. Argue that the result establishes the scaling property.

$$\delta(k[x - x_o]) \approx |\,k\,|^{-1}\,\delta(x - x_o)$$

9.) **Advanced Scaling Property**: Prove the advanced scaling property. As always, the functions $f(x)$ and $g(x)$ are assumed continuous and continuously differentiable.

$$\int_a^b f(x)\;\delta\left[g(x) - g(x_0)\right]\,dx = \begin{cases} f(x_0)/\left|g'(x_0)\right| & \text{if } x_0 \in (a,b) \\ 0 & \text{if } x_0 \notin [a,b] \end{cases} \quad \text{where } g'(x) = \tfrac{dg}{dx}$$

10.) Consider the electric field due to a uniformly charges sphere of radius n^{-1} and total charge Q. Compute the divergence of the electric field for $r < n^{-1}$ and for $r > n^{-1}$. Take the limit that $n \to \infty$ with constant charge to develop a 3D delta function. Reduce to an identity for div ($\vec{r}/r^3$). Recall that div ($\vec{r}/r^3$) = - $\nabla^2\left(\frac{1}{r}\right)$.

11.) Consider the magnetic field due to a long straight wire with uniform current density, total current I_o and radius $r = n^{-1}$. Compute the curl of the magnetic field for $r < n^{-1}$ and for $r > n^{-1}$. Take the limit that $n \to \infty$ while maintaining constant current I_o. Compare with the definition of a 2D delta function. Evaluate curl of $\left(\hat{k}\times\vec{r}\right)/r^2$ using deltas as appropriate.

12.) Show that ${}^{Yukawa}D_n^{(3)}(\vec{r}) = \left(\dfrac{n^2}{2\pi}\right)\dfrac{e^{-n^2 r^2}}{r}$ is sequence that represents $\delta^3(\vec{r})$ in spherical coordinates. Hence ${}^{Yukawa}D_n^{(3)}(\vec{r}-\vec{r}\,') = \left(\dfrac{n^2}{2\pi}\right)\dfrac{e^{-n^2|\vec{r}-\vec{r}\,'|^2}}{|\vec{r}-\vec{r}\,'|} \to \delta^3(\vec{r}-\vec{r}\,')$.

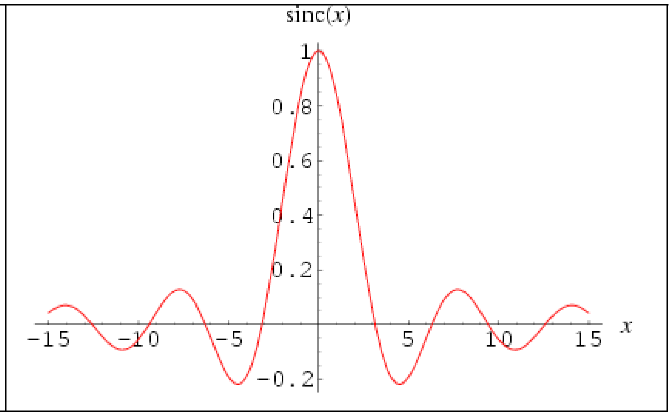
13.) Show that ${}^{gamma}D_n^{(3)}(\vec{r}) = \left(\dfrac{n^3}{8\pi}\right)e^{-nr}$ is sequence that represents $\delta^3(\vec{r})$ in spherical coordinates.

Hence $^{gamma}D_n^{(3)}(\vec r - \vec r\,') = \left(\frac{n^3}{8\pi}\right) e^{-n|\vec r - \vec r\,'|} \to \delta^3(\vec r - \vec r\,')$.

More Alternatives:

$$^{?}D_n(x) = \left(\frac{1}{2\pi}\right)\left(\frac{\sin([n+\tfrac{1}{2}]x)}{\sin(\tfrac{1}{2}x)}\right)$$

See: mathworld.wolfram.com/



14.) Show that $x\,\dfrac{d\,\delta(x)}{dx} = -\delta(x)$.

15.) Show that $\delta((x-a)(x-b)) = \dfrac{1}{|a-b|}\left[\delta(x-a)+\delta(x-b)\right]$. (**Manogue – Oregon State)**

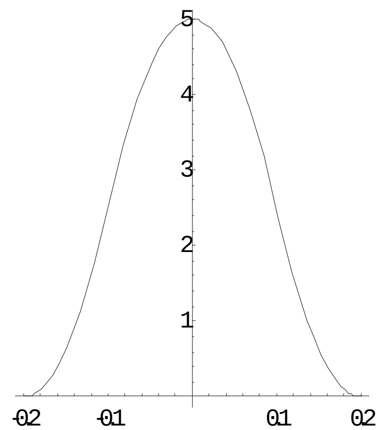
16.) Argue that $\dfrac{1}{2}\left(\dfrac{d^2}{dx^2}|x|\right) = \delta(x)$. (**Manogue – Oregon State)**

17.) **Advanced Scaling Property II**: Argue that $\delta(g(x) - g(x_o)) \approx \delta\left(\left.\frac{dg}{dx}\right|_{x_o}(x - x_0)\right)$, and that the advanced scaling property follows from the scaling property.

18.) Argue that $\delta(x^2 - 4) = \delta((x - 2)(x + 2)) \approx \delta((-4)(x + 2)) + \delta(4(x - 2))$.

19.) Consider the delta representation:

$$^{17}D_n(x) = \begin{cases} n^3\,(x + \tfrac{2}{n})^2/4 & \text{for } -\tfrac{2}{n} < x < -\tfrac{1}{n} \\ (2n - n^3 x^2)^2/4 & \text{for } -\tfrac{1}{n} < x < +\tfrac{1}{n} \\ n^3\,(\tfrac{2}{n} - x)^2/4 & \text{for } +\tfrac{1}{n} < x < +\tfrac{2}{n} \end{cases}$$



Show that $\frac{d^2}{dx^2}\left[{}^{17}D_n(x)\right] \approx n\ \{½\ n\ [\delta(x + {}^3/_{2n}) - \delta(x + {}^1/_{2n})] - ½\ n\ [\delta(x + {}^1/_{2n}) - \delta(x + {}^3/_{2n})]\ \}$ so that

$$\int_{-\infty}^{\infty} f(x)\, \frac{d^2}{dx^2}\big(\delta(x - x_0)\big)\, dx \approx \frac{\dfrac{f(x_0 + \frac{3}{2n}) - f(x_0 + \frac{1}{2n})}{n^{-1}} - \dfrac{f(x_0 - \frac{1}{2n}) - f(x_0 - \frac{3}{2n})}{n^{-1}}}{2\,n^{-1}} = \left.\frac{d^2 f}{dx^2}\right|_{x_0}$$

Use integration by parts to formally evaluate $\int_{-\infty}^{\infty} f(x)\, \frac{d^2}{dx^2}\big(\delta(x - x_0)\big)\, dx$. Compare the results and discuss your findings.

20. Evaluate $\int_0^{20} \delta(x-5)\,\delta(x-7)\,dx$. Justify your result first by focusing on the region around the argument zero of each delta and treating the *other* delta function as just a function in that region. Generate a second justification by preparing a sketch using the ${}^1D_n(x)$ (tall rectangle) model of the delta function for $n = 4$. Sketch he product function ${}^1D_n(x\ \text{-}5)\ {}^1D_n(x - 7)$. Comment.

21. Evaluate $\sum_{m=1}^{m=20} \delta_{m5}\ \delta_{m7}$.

22. Provide a justification for each evaluation. As a step, identify the values of the integration variable for which argument zeroes occur. Note the ones that are in the range of the integration.

Evaluate: a.) $3\sqrt{2}\int_0^{2\pi} \cos(\theta)\,\delta(3[\theta-\pi])\,d\theta$; b.) $2\sqrt{2}\int_0^{2} e^{x^2/2}\ \ \delta(x^2-2)\,dx$,

c.) $\int_0^4 \left(4\,x^{-2}\right)\ \frac{d[\delta(x-2)]}{dx}\,dx$; d.) $\int_{0.25}^{2.25} \cos(x)\ \ \delta(\sin(\pi x))\,dx$

23. a.) Consider $F(x) = \int_{-\infty}^{x} \delta(x' - a)\, dx'$. Give the value of $F(x)$ for all x.

b.) The Heaviside (unit-step) function $\Theta(x - x_o)$ is zero for $x < x_o$ and equals one for $x > x_o$. Describe the derivative of $\Theta(x - x_o)$. Express $F(x)$ defined in part a.) in terms of a Heaviside function.

24. Gauss's Law has the form $\vec{\nabla} \cdot \vec{E} = \rho(\vec{r}) / \varepsilon_o$. For a problem with spherical symmetry, the electric field has the form $\vec{E}(\vec{r}) = E_r(r)\,\hat{r}$ and, of course, $\rho(\vec{r}) = \rho(r)$. Substituting, it follows that:

$$\frac{\partial}{\partial r}\left(r^2\, E_r(r)\right) = r^2 \rho(r) / \varepsilon_o$$

a.) Use the equation above to find E_r for all r given the charge density:

$$\rho(\vec{r}) = \begin{cases} \rho_o & (r < R) \\ 0 & (r > R) \end{cases}$$

Argue that $E_r(r = 0)$ should be zero. Give an alternative representation in terms of the total charge of the sphere q_{tot} and eliminate ρ_o. Validate your final result by choosing a spherical surface with radius r just greater than R and verifying that $\oint \vec{E} \cdot \hat{n}\; dA = q_{enc} / \varepsilon_o$ where $q_{enc} = \rho_o \frac{4\pi R^3}{3}$.

b.) One might decide to integrate from ∞ to r. The equation would be:

$$\left((r')^2\, E_r(r')\right)\Big|_{r'=r} - \left((r')^2\, E_r(r')\right)\Big|_{r'=\infty} = \int_{\infty}^{r} (r')^2 \rho(r') / \varepsilon_o \; dr'$$

which is difficult to interpret unless one assumes the result that, as $r \to \infty$, $E_r \to \dfrac{q_{total}}{4\pi\, \varepsilon_o r^2} + \mathcal{O}(r^{-3})$

where q_{total} is the total charge of the distribution and $\rho(r) \equiv 0$ for $r > R$, a finite positive value.* The term $\mathcal{O}(r^{-3})$ represents terms of order r^{-3} and higher; that is: the higher multi-poles with contributions that vanish as fast as r^{-3} or faster as $r \to \infty$. Show that the equation becomes:

$$\left((r')^2\, E_r(r')\right)\Big|_{r'=r} - \frac{q_{total}}{4\pi\, \varepsilon_o} = \int_{\infty}^{r} (r')^2 \rho(r') / \varepsilon_o \; dr'.$$

Use this alternative approach to find $E_r(r)$ as you repeat part a.

* At great distances from a charge distribution of finite extent, the field due to the distribution approaches that of a point charge with the total charge of the distribution plus contributions that vanish as fast as r^{-3} or faster as $r \to \infty$. See the multi-poles handout for more detail. For the case just

above there are no contributions of the form $\mathcal{O}(r^{-3})$. The higher order multi=poles have more complex angular variations. Only the monopole, the net charge, has a potential that is isotropic. An indeterminate from such as $\left((r')^2 E_r(r')\right)\Big|_{r'=\infty}$ is defined as: $\underset{r'\to\infty}{Limit}\left((r')^2 E_r(r')\right) = \frac{q_{total}}{4\pi\varepsilon_o}$.

25. Gauss's Law has the form $\vec{\nabla}\cdot\vec{E} = \rho(\vec{r})/\varepsilon_O$. For a problem with spherical symmetry, the electric field has the form $\vec{E}(\vec{r}) = E_r(r)\,\hat{r}$ and, of course, $\rho(\vec{r}) = \rho(r)$. Substituting, it follows that:

$$\frac{\partial}{\partial r}\left(r^2 E_r(r)\right) = r^2\rho(r)/\varepsilon_O$$

a.) In problem 3, it was shown that $\rho(\vec{r}) = \sigma\ \ \delta(r-R)$ represented the charge density for a uniform spherical shell of surface charge density σ and radius R centered o the origin. Find E_r for all r given this charge density. Use: $\frac{\partial}{\partial r}\left(r^2 E_r(r)\right) = r^2\rho(r)/\varepsilon_O$ and integrate from 0 to r.

b.) One might decide to integrate from ∞ to r. The equation would be:

$$\left((r')^2 E_r(r')\right)\Big|_{r'=r} - \left((r')^2 E_r(r')\right)\Big|_{r'=\infty} = \int_\infty^r (r')^2\rho(r')/\varepsilon_O\, dr'$$

which is difficult to interpret unless one assumes the result that, as $r\to\infty$, $E_r \to \frac{q_{total}}{4\pi\varepsilon_o r^2} + \mathcal{O}(r^{-3})$

where q_{total} is the total charge of the distribution and $\rho(r) \equiv 0$ got $r > K$, some finite positive value.* The term $\mathcal{O}(r^{-3})$ represents terms of order r^{-3} and higher; that is: contributions that vanish as fast as r^{-3} or faster as $r\to\infty$. Show that the equation becomes: $\left((r')^2 E_r(r')\right)\Big|_{r'=r} - \frac{q_{total}}{4\pi\varepsilon_o} = \int_\infty^r (r')^2\rho(r')/\varepsilon_O\, dr'$.

Use this alternative approach to find $E_r(r)$ as you repeat part a.

* At great distances from a charge distribution of finite extent, the field due to the distribution approaches that of a point charge with the total charge of the distribution plus contributions that vanish as fast as r^{-3} or faster as $r\to\infty$. See the multipoles handout for more detail. For the case just above there are no contributions of the form $\mathcal{O}(r^{-3})$.

26.) Evaluate: a.) $\int_0^{2\pi}\cos(\theta)\,\delta(\theta-0.75\pi)\,d\theta$

b.) $3\sqrt{2}\int_0^{2\pi}\cos(\theta)\,\delta(3[\theta-\pi])\,d\theta$;

c.) $2\sqrt{2}\int_0^2 e^{x^2/2}\ \delta(x^2-2)\,dx$,

d.) $\int_0^4 \left(4x^{-2}\right)\ \frac{d[\delta(x-2)]}{dx}\,dx$

e.) $\int_{0.25}^{2.25}\cos(x)\ \ \delta(\sin(\pi x))\,dx$

f.) $\int_0^1 x^2\ \ \delta(x-2)\,dx$,

g.) $\int_0^5 \cos(\pi x)\ \delta(x-1)\,\delta(x-4)\,dx$

27. Evaluate the following expressions.

a.) $\int_0^{2\pi}\cos(\theta)\,\delta(\theta-\pi)\,d\theta$ b.) $\int_0^{2\pi}\cos(\theta)\,\delta(\pi-\theta)\,d\theta$; c.) $\int_0^{2\pi}\cos(\theta)\,\delta(3[\theta-\pi])\,d\theta$;

d.) $\int_{-5}^5 x^2\ \ \delta(2\ x^2-8)\,dx$, e.) $\int_0^4 x^3\ \frac{d[\delta(x-3)]}{dx}\,dx$ f.) $\int_{0.25}^{1.75}\cos(\pi x/3)\ \ \delta(\cos(\pi x))\,dx$

28. Evaluate: a.) $\int_0^5 x^3\,\delta(x-2)\,dx$; b.) $\int_0^5 x^3\,\delta(2-x)\,dx$; c.) $\int_{-3}^5 x^3\,\delta(4[x+2])\,dx$;

d.) $\sqrt{8}\int_{-5}^5 e^{-x^2/2}\ \ \delta(2\ x^2-4)\,dx$; e.) $\int_0^4 sin(\pi x)\ \frac{d[\delta(x-3)]}{dx}\,dx$; f.) $\int_{-0.25}^{0.25}\cos(\pi x/3)\ \ \delta(\cos(\pi x))\,dx$.

29. Evaluate: a.) $\int_0^{2\pi}\sin(2\theta)\,\delta(\theta-0.75\pi)\,d\theta$ b.) $4\sqrt{2}\int_0^{2\pi}\cos(\theta)\,\delta(4\theta-\pi)\,d\theta$;

c.) $2\sqrt{2}\int_0^2 e^{x^2/2}\ \delta(x^2-2)\,dx$, d.) $\int_0^4\left(4x^{-2}\right)\ \frac{d[\delta(x-2)]}{dx}\,dx$

e.) $\int_{0.25}^{2.25}\cos(\pi x/6)\ \ \delta(\sin(\pi x))\,dx$ f.) $\int_0^1 x^2\ \ \delta(x-2)\,dx$

30. The Green function for the Laplace operators. ***** ADD uniformly charged sphere, …

References:

1. The Wolfram web site: mathworld.wolfram.com/

2. K. F. Riley, M. P. Hobson and S. J. Bence, ***Mathematical Methods for Physics and Engineering***, 2^{nd} Ed., Cambridge, Cambridge UK (2002).

3. T. Dray and C. A. Manogue, Oregon State

****

******* New extensive edit**

Proposed New Evaluation Rule: In the case that the argument of the delta function is itself a function and it is possible to choose that function as the new integration variable, this proposed method might work.

$$\int_a^b f(x)\ \delta(u(x)-u(x_0))\ \ dx \ \ \to \int_{u(a)}^{u(b)} g(u)\ \delta(u-u(x_0))\ \ du$$

Let $\{x_1, \ldots, x_j, \ldots\}$ be the set of values for which $u(x) - u(x_o) = 0$ causing the delta function explode.

$$\int_a^b f(x)\ \delta(u(x)-u(x_0))\ \ dx \ \ = \sum_{x_i} g[u(x_i)]\left\{\frac{f(x_i)}{\left|f(x_i)\right|}\right\}$$

where the sum is over the values x_i the corresponding to the zeros of the delta function argument. In the original expression, the delta function provides positive weight to the value of $f(x)$ at each value x_i. Hence the sign of $f(x_i)$ is to be assigned to the contribution at that zero of the argument of the delta function. The factor of $f(x_i)$ divided by its own absolute value supplies that sign.

Exercise: Solve for the form of $g(u)$ that results after the change of variable discussed above. Does your result support the claim that the relation above is just a restatement of the advanced scaling property? Explain.