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Stability Analysis of a Class of Fractional-order Neural Networks

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Abstract

In this paper, the problems of the existence and uniqueness of solutions and stability for a class of fractional-order neural networks are studied by using Banach fixed point principle and analysis technique, respectively. A sufficient condition is given to ensure the existence and uniqueness of solutions and uniform stability of solutions for fractional-order neural networks with variable coefficients and multiple time delays. The obtained results improve and extend some previous works to some extent, and they are easy to check in practice. An illustrative example is presented to show the validity and application of the proposed results.

Keywords: *neural networks, stability, fractional-order, time-varying coefficients, delays*

1. Introduction

In recent years, much attention has been focused on the study of fractional calculus. As a branch of mathematical analysis and an ongoing topic, fractional calculus deals with derivatives and integrals of arbitrary non-integer order (rational, irrational or even complex). The applications of fractional calculus have been found in many areas such as chemistry [1], optics [2], biology [3], economics [4], finances [5], electricity [6], mechanics [7], physics [8], and control theory [9]. The one of the main reason for the extensive applications of fractional calculus is that fractional derivatives provide an efficient and excellent instrument for the description of memory and hereditary properties of various materials and processes compared to integral-order derivatives. So there are two advantages in models of fractional-order, one is more degrees of freedom in the models, the other is “memory” in the models. Neural networks have been proven to be very efficient at handling a wide range of engineering application [10-12]. Nowadays, fractional calculus has been used in modeling artificial neural networks; the fractional-order formulation of neural network models is also justified by research results about biological neurons [13-16]. Especially, the authors emphasized the utility of developing and studying fractional-order mathematical models of neural network in [14].

The problem of stability is a very fundamental and crucial issue for fractional-order neural networks, however, due to the high complexity of fractional calculus, it has been investigated and discussed only in some recent literature, and only very few relevant results have been obtained, for instance [17] and [18]. In [17], stability and multi-stability of fractional-order Hopfield neural networks were discussed, but in the case of no time delay and constant coefficient. In [18], a sufficient condition ensuring uniform stability and the existence, uniqueness of equilibrium point was established for a class of fractional-order neural networks with single constant delay, but the initial conditions was assumed to be zero initial conditions, and without considering variable coefficients. As we all know, there is no related work on the stability analysis of fractional-order neural networks with variable coefficients up till now, although some excellent results concerning the stability of integer-order neural networks with variable coefficients have been obtained [19-24]. In addition, it is well known that communication delays are ubiquitous in many real world phenomena, and often become sources of instability. Motivated by the above discussions, this paper is devoted to presenting a theoretical stability analysis for a class of fractional-order neural networks with variable coefficients and multiple time delays.

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The paper is structured as follows. In Section 2, some basic definitions and lemmas of fractional calculus are given. In Section 3, the description of fractional-order neural networks with variable coefficients and multiple time delays is presented, and the main results are derived. In Section 4, an example is used to illustrate the results obtained in this paper. Some conclusions are drawn in Section 5.

2. Preliminaries

We introduce the space $\Omega = (C([0,T], R^n), \|\cdot\|)$ as a Banach space, where $C([0,T], R^n)$ is the class of all continuous column n-vectors function. For $\xi \in C([0,T], R^n)$, the norm is defined by $\|\xi\| = \sum_{i=1}^{n} \sup_t \{e^{-Nt} |\xi_i(t)|\}$. Besides, for a matrix $A = (a_{ij}(t))_{n\times n}$, we define the norm $\|A\| = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \sup_{t,\forall j} |a_{ij}(t)|$.

Definition 1. The fractional order integral of a function $f(t)$ of order $\alpha \in R^+$ is defined by:

$$I_{t_0}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \tag{1}$$

Where $\Gamma(\cdot)$ is the gamma function defined as:

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt. \tag{2}$$

Definition 2. The Caputo fractional derivative D^{α} of order α of a function $f(t)$ is given by:

$$D_{t_0}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \tag{3}$$

Where $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of the number α.

Lemma 1 [25]**.** If the Caputo fractional derivative $D_{t_0}^{\alpha} f(t)$ $(n-1 \le \alpha < n)$ is integrable, then:

$$I_{t_0}^{\alpha} D_{t_0}^{\alpha} f(t) = f(t) - \sum_{i=0}^{n-1} \frac{f^{(i)}(t_0)}{i!} (t-t_0)^i. \tag{4}$$

Especially, for $0 < \alpha < 1$, one can obtain:

$$I_{t_0}^{\alpha} D_{t_0}^{\alpha} f(t) = f(t) - f(t_0). \tag{5}$$

3. Research Method

Consider the following neural network model with variable coefficients and multiple time delays:

$$\begin{aligned} D^{\alpha} x_i(t) = {} & -c_i(t) x_i(t) + \sum_{j=1}^{n} a_{ij}(t) f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t) f_j(x_j(t-\tau_{ij})) \\ & + I_i(t), \quad t \in [0,T], \end{aligned} \tag{6}$$

Where $T < +\infty$; D^{α} denotes Caputo fractional-order derivative of order α $(0 < \alpha < 1)$; $i = 1, 2, \ldots, n$ and n corresponds to the number of units in a neural network; $x_i(t)$ denotes the state of the i th unit at time t; $f_j(x_j(t))$ denotes the activation functions of the j th unit at time t

; $c_i(t) > 0$ corresponds to the rate with which the i th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs at time t; $a_{ij}(t)$ and $b_{ij}(t)$ represent the connection strength of the j th unit on the i th unit at time t and $t-\tau_{ij}$, respectively; $I_i(t)$ denotes the external inputs at time t; τ_{ij} corresponds to the transmission delay along the axon of the j th unit, and $0 \le \tau_{ij} \le \tau = \max\{\tau_{ij} \mid i, j = 1, 2, \ldots, n\}$.

The initial conditions associated with (6) are of the form:

$$x_i(t) = \phi_i(t),\ t \in [-\tau, 0],\ i = 1, 2, \ldots, n, \tag{7}$$

Where $\phi_i(t) \in C([-\tau, 0], R)$, and the norm of an element in $C([-\tau, 0], R^n)$ is $\|\phi\| = \sum_{i=1}^{n} \sup_{t\in[-\tau,0]}\{e^{-Nt} \mid \phi_i(t) \mid\}$.

Throughout this paper, we impose the following assumptions to obtain our results.

Assumption 1. $c_i(t), a_{ij}(t), b_{ij}(t)$ and $I_i(t)$ are continuous on $[0, T]$.

Assumption 2. The activation functions f_j are Lipschitz continuous, i.e., there exist positive constants L_j such that:

$$| f_j(u) - f_j(v) | \le L_j | u - v |, \tag{8}$$

For all $u, v \in R$, where $j = 1, 2, \ldots n$.

For convenience, we introduce the following notation related to model (6):

$\| A \| = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \sup_{t,\forall j} | a_{ij}(t) |$, $\| B \| = \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} \sup_{t,\forall j} | b_{ij}(t) |$, $c^* = \max\{c_1, c_2, \ldots, c_n\} = \max\{ \sup_t | c_1(t) |, \sup_t | c_2(t) |, \ldots, \sup_t | c_n(t) | \}$, $L = \max\{L_1, L_2, \ldots, L_n\}$.

3.1. Existence and Uniqueness

Theorem 1. Assume assumption 1 and 2 hold, the system (6) has a unique solution $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in C([0, T], R^n)$ satisfying the initial condition (7).

Proof. According to the properties of the fractional calculus, one can obtain a solution of (6) in the form of the equivalent Volterra integral equation:

$$\begin{aligned} x_i(t) &= \phi_i(0) + I^\alpha D^\alpha x_i(t) \\ &= \phi_i(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [-c_i(s) x_i(s) + \sum_{j=1}^{n} a_{ij}(s) f_j(x_j(s)) \\ &\quad + \sum_{j=1}^{n} b_{ij}(s) f_j(x_j(s - \tau_{ij})) + I_i(s)] ds, \end{aligned} \tag{9}$$

Where $t \in [0, T]$.

We transform the problem (9) into a fixed problem. Consider a mapping defined by:

$$F : R^n \to R^n, \tag{10}$$

Where $Fx = (F_1 x_1, F_2 x_2, \ldots, F_n x_n)^T$, and F_i is defined as follow:

$$\begin{aligned} F_i x_i(t) &= \phi_i(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [-c_i(s) x_i(s) \\ &\quad + \sum_{j=1}^{n} (a_{ij}(s) + b_{ij}(s)) f_j(x_j(s)) + I_i(s)] ds. \end{aligned} \tag{11}$$

For any two different functions $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$, $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^T$, we have:

$$\begin{aligned}
&| F_i x_i(t) - F_i y_i(t) | \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [c_i(s) | x_i(s) - y_i(s) | + \sum_{j=1}^{n} (| a_{ij}(s) | + | b_{ij}(s) |) | f_j(x_j(s)) - f_j(y_j(s)) |] ds \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [c_i(s) | x_i(s) - y_i(s) | + \sum_{j=1}^{n} (| a_{ij}(s) | + | b_{ij}(s) |) L_j | x_j(s) - y_j(s) |] ds \\
&\le \frac{1}{\Gamma(\alpha)} \sup_t | c_i(t) | \int_0^t (t-s)^{\alpha-1} | x_i(s) - y_i(s) | ds \\
&\quad + \frac{1}{\Gamma(\alpha)} (\sup_{t,\forall j} | a_{ij}(t) | + \sup_{t,\forall j} | b_{ij}(t) |) L \int_0^t (t-s)^{\alpha-1} \sum_{j=1}^{n} | x_j(s) - y_j(s) | ds \\
&= \frac{1}{\Gamma(\alpha)} c_i \int_0^t (t-s)^{\alpha-1} | x_i(s) - y_i(s) | ds \\
&\quad + \frac{1}{\Gamma(\alpha)} (a_i + b_i) L \int_0^t (t-s)^{\alpha-1} \sum_{j=1}^{n} | x_j(s) - y_j(s) | ds,
\end{aligned} \tag{12}$$

Which implies that:

$$\begin{aligned}
&e^{-Nt} | F_i x_i(t) - F_i y_i(t) | \\
&\le \frac{1}{\Gamma(\alpha)} c_i e^{-Nt} \int_0^t (t-s)^{\alpha-1} | x_i(s) - y_i(s) | ds \\
&\quad + \frac{1}{\Gamma(\alpha)} (a_i + b_i) L e^{-Nt} \int_0^t (t-s)^{\alpha-1} \sum_{j=1}^{n} | x_j(s) - y_j(s) | ds \\
&= \frac{1}{\Gamma(\alpha)} c_i \int_0^t (t-s)^{\alpha-1} e^{-N(t-s)} e^{-Ns} | x_i(s) - y_i(s) | ds \\
&\quad + \frac{1}{\Gamma(\alpha)} (a_i + b_i) L \int_0^t (t-s)^{\alpha-1} e^{-N(t-s)} e^{-Ns} \sum_{j=1}^{n} | x_j(s) - y_j(s) | ds \\
&\le c_i \sup_t \{ e^{-Nt} | x_i(t) - y_i(t) | \} \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-N(t-s)} ds \\
&\quad + (a_i + b_i) L \sum_{j=1}^{n} \sup_t \{ e^{-Nt} | x_j(t) - y_j(t) | \} \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-N(t-s)} ds \\
&\le \tfrac{c_i}{N^\alpha} \sup_t \{ e^{-Nt} | x_i(t) - y_i(t) | \} + \tfrac{(a_i + b_i) L}{N^\alpha} \| x(t) - y(t) \| .
\end{aligned} \tag{13}$$

Obviously, we have:

$$\begin{aligned}
&\| Fx(t) - Fy(t) \| \\
&= \sum_{i=1}^{n} \sup_t \{ e^{-Nt} | F_i x_i(t) - F_i y_i(t) | \} \\
&\le \sum_{i=1}^{n} \tfrac{c_i}{N^\alpha} \sup_t \{ e^{-Nt} | x_i(t) - y_i(t) | \} + \sum_{i=1}^{n} \tfrac{(a_i + b_i) L}{N^\alpha} \| x(t) - y(t) \| \\
&\le (\tfrac{c^*}{N^\alpha} + \tfrac{(\| A \| + \| B \|) L}{N^\alpha}) \| x(t) - y(t) \| .
\end{aligned} \tag{14}$$

Now, choose N large enough such that $c^* + (\| A \| + \| B \|) L < N^\alpha$, then we have:

$$\| Fx(t) - Fy(t) \| < \| x(t) - y(t) \| . \tag{15}$$

Therefore the mapping F is a contraction mapping. As a consequence of the Banach fixed point theorem, the problem (9) has a unique fixed point, so that we conclude that system (6) has a unique solution, which complete the proof of the theorem.

3.2. Stability

Definition 5. The solution of system (6) will be called stable if for any $\varepsilon > 0$, $t_0 \ge 0$, there exists a corresponding value $\delta(\varepsilon, t_0) > 0$ such that $\| y(t, t_0, \varphi) - x(t, t_0, \phi) \| < \varepsilon$ for $t \ge t_0$ as soon as initial conditions satisfy $\| \varphi(t) - \phi(t) \| < \delta(\varepsilon, t_0)$. The solution of (6) will be called uniformly stable if the above δ can be chosen independently of t_0 : $\delta(\varepsilon, t_0) \equiv \delta(\varepsilon)$.

Theorem 2. If assumption 1 and 2 are satisfied, the solution of system given by (6) satisfying initial condition (7) is uniformly stable.

Proof. Let $x(t)=(x_1(t),x_2(t),\dots,x_n(t))^T$ and $y(t)=(y_1(t),y_2(t),\dots,y_n(t))^T$ be two solutions of (6) with the different initial condition $x_i(t)=\phi_i(t)$, $y_i(t)=\varphi_i(t)$, $i=1,2,\dots,n$. Then for $t\in[0,T]$, we have:

$$\begin{aligned} D^\alpha(y_i(t)-x_i(t)) &= -c_i(t)(y_i(t)-x_i(t))+\sum_{j=1}^n a_{ij}(t)(f_j(x_j(t))-f_j(y_j(t))) \\ &\quad+\sum_{j=1}^n b_{ij}(t)(f_j(x_j(t-\tau_{ij}))-f_j(y_j(t-\tau_{ij}(t)))), \end{aligned} \tag{16}$$

Which is equivalent to the nonlinear Volterra integral equation, given by the following form:

$$\begin{aligned} & y_i(t)-x_i(t) \\ &= \varphi_i(0)-\phi_i(0)+I^\alpha[-c_i(t)(y_i(t)-x_i(t))+\sum_{j=1}^n a_{ij}(t)(f_j(y_j(t))-f_j(x_j(t))) \\ &\quad+\sum_{j=1}^n b_{ij}(t)(f_j(y_j(t-\tau_{ij}))-f_j(x_j(t-\tau_{ij})))] \\ &= \varphi_i(0)-\phi_i(0)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[-c_i(s)(y_i(s)-x_i(s)) \\ &\quad+\sum_{j=1}^n a_{ij}(s)(f_j(y_j(s))-f_j(x_j(s))) \\ &\quad+\sum_{j=1}^n b_{ij}(s)(f_j(y_j(s-\tau_{ij}))-f_j(x_j(s-\tau_{ij})))]ds. \end{aligned} \tag{17}$$

From (17), we get:

$$\begin{aligned} & e^{-Nt}|y_i(t)-x_i(t)| \\ &\le e^{-Nt}|\varphi_i(0)-\phi_i(0)|+\frac{1}{\Gamma(\alpha)}e^{-Nt}\int_0^t(t-s)^{\alpha-1}[c_i(s)|y_i(s)-x_i(s)| \\ &\quad+\sum_{j=1}^n|a_{ij}(s)||f_j(y_j(s))-f_j(x_j(s))|+\sum_{j=1}^n|b_{ij}(s)||f_j(y_j(s-\tau_{ij}))-f_j(x_j(s-\tau_{ij}))|]ds \\ &\le e^{-Nt}|\varphi_i(0)-\phi_i(0)|+\frac{1}{\Gamma(\alpha)}e^{-Nt}\sup_t c_i(t)\int_0^t(t-s)^{\alpha-1}|y_i(s)-x_i(s)|ds \\ &\quad+\frac{1}{\Gamma(\alpha)}e^{-Nt}\sup_{t,\forall j}|a_{ij}(t)|\int_0^t(t-s)^{\alpha-1}\sum_{j=1}^n|f_j(y_j(s))-f_j(x_j(s))|ds \\ &\quad+\frac{1}{\Gamma(\alpha)}e^{-Nt}\sup_{t,\forall j}|b_{ij}(t)|\int_0^t(t-s)^{\alpha-1}\sum_{j=1}^n|f_j(y_j(s-\tau_{ij}))-f_j(x_j(s-\tau_{ij}))|ds \\ &\le e^{-Nt}|\varphi_i(0)-\phi_i(0)|+\frac{1}{\Gamma(\alpha)}e^{-Nt}c_i\int_0^t(t-s)^{\alpha-1}|y_i(s)-x_i(s)|ds \\ &\quad+\frac{1}{\Gamma(\alpha)}e^{-Nt}a_iL\int_0^t(t-s)^{\alpha-1}\sum_{j=1}^n|y_j(s)-x_j(s)|ds \\ &\quad+\frac{1}{\Gamma(\alpha)}e^{-Nt}b_iL\int_0^t(t-s)^{\alpha-1}\sum_{j=1}^n|y_j(s-\tau_{ij})-x_j(s-\tau_{ij})|ds \\ &\le e^{-Nt}|\varphi_i(0)-\phi_i(0)|+\frac{1}{\Gamma(\alpha)}c_i\int_0^t(t-s)^{\alpha-1}e^{-N(t-s)}e^{-Ns}|y_i(s)-x_i(s)|ds \\ &\quad+\frac{1}{\Gamma(\alpha)}a_iL\int_0^t(t-s)^{\alpha-1}\sum_{j=1}^n e^{-N(t-s)}e^{-Ns}|y_j(s)-x_j(s)|ds \\ &\quad+\frac{1}{\Gamma(\alpha)}b_iL\int_0^t(t-s)^{\alpha-1}\sum_{j=1}^n e^{-N(t-s+\tau_{ij})}e^{-N(s-\tau_{ij})}|y_j(s-\tau_{ij})-x_j(s-\tau_{ij})|ds \\ &\le e^{-Nt}|\varphi_i(0)-\phi_i(0)|+\frac{1}{\Gamma(\alpha)}c_i\int_0^t(t-s)^{\alpha-1}e^{-N(t-s)}e^{-Ns}|y_i(s)-x_i(s)|ds \\ &\quad+\frac{1}{\Gamma(\alpha)}a_iL\int_0^t(t-s)^{\alpha-1}\sum_{j=1}^n e^{-N(t-s)}e^{-Ns}|y_j(s)-x_j(s)|ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)} b_i L \sum_{j=1}^{n} \int_0^t (t-s)^{\alpha-1} e^{-N(t-s+\tau_{ij})} e^{-N(s-\tau_{ij})} \mid y_j(s-\tau_{ij}) - x_j(s-\tau_{ij}) \mid ds \\
&\le e^{-Nt} \mid \varphi_i(0) - \phi_i(0) \mid + \frac{1}{\Gamma(\alpha)} c_i \int_0^t (t-s)^{\alpha-1} e^{-N(t-s)} e^{-Ns} \mid y_i(s) - x_i(s) \mid ds \\
&+\frac{1}{\Gamma(\alpha)} a_i L \int_0^t (t-s)^{\alpha-1} \sum_{j=1}^{n} e^{-N(t-s)} e^{-Ns} \mid y_j(s) - x_j(s) \mid ds \\
&+\frac{1}{\Gamma(\alpha)} b_i L \sum_{j=1}^{n} \int_0^{\tau_{ij}} (t-s)^{\alpha-1} e^{-N(t-s+\tau_{ij})} e^{-N(s-\tau_{ij})} \mid y_j(s-\tau_{ij}) - x_j(s-\tau_{ij}) \mid ds \\
&+\frac{1}{\Gamma(\alpha)} b_i L \sum_{j=1}^{n} \int_{\tau_{ij}}^t (t-s)^{\alpha-1} e^{-N(t-s+\tau_{ij})} e^{-N(s-\tau_{ij})} \mid y_j(s-\tau_{ij}) - x_j(s-\tau_{ij}) \mid ds \\
&\le \sup_{t\in[-\tau,0]} \{e^{-Nt} \mid \varphi_i(t) - \phi_i(t) \mid\} + c_i \sup_t \{e^{-Nt} \mid y_i(t) - x_i(t) \mid\} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-N(t-s)} ds \\
&+ a_i L \sum_{j=1}^{n} \sup_t \{e^{-Nt} \mid y_j(t) - x_j(t) \mid\} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-N(t-s)} ds \\
&+ b_i L \sum_{j=1}^{n} \left[\sup_{t\in[0,\tau_{ij}]} \{e^{-N(t-\tau_{ij})} \mid y_j(t-\tau_{ij}) - x_j(t-\tau_{ij}) \mid\} \frac{1}{\Gamma(\alpha)} \int_0^{\tau_{ij}} (t-s)^{\alpha-1} e^{-N(t-s+\tau_{ij})} ds \right] \\
&+ b_i L \sum_{j=1}^{n} \left[\sup_{t\in[\tau_{ij},t]} \{e^{-N(t-\tau_{ij})} \mid y_j(t-\tau_{ij}) - x_j(t-\tau_{ij}) \mid\} \frac{1}{\Gamma(\alpha)} \int_{\tau_{ij}}^t (t-s)^{\alpha-1} e^{-N(t-s+\tau_{ij})} ds \right] \\
&\le \sup_{t\in[-\tau,0]} \{e^{-Nt} \mid \varphi_i(t) - \phi_i(t) \mid\} + \tfrac{c_i}{N^\alpha} \sup_t \{e^{-Nt} \mid y_i(t) - x_i(t) \mid\} \\
&+ \tfrac{a_i L}{N^\alpha} \| y(t) - x(t) \| + \tfrac{b_i L}{N^\alpha} \sum_{j=1}^{n} \sup_{t\in[-\tau,0]} \{e^{-Nt} \mid \varphi_j(t) - \phi_j(t) \mid\} \\
&+ \tfrac{b_i L}{N^\alpha} \sum_{j=1}^{n} \sup_t \{e^{-Nt} \mid y_j(t) - x_j(t) \mid\} \\
&\le \sup_{t\in[-\tau,0]} \{e^{-Nt} \mid \varphi_i(t) - \phi_i(t) \mid\} + \tfrac{c_i}{N^\alpha} \sup_t \{e^{-Nt} \mid y_i(t) - x_i(t) \mid\} \\
&+ \tfrac{(a_i+b_i)L}{N^\alpha} \| y(t) - x(t) \| + \tfrac{b_i L}{N^\alpha} \| \varphi(t) - \phi(t) \| .
\end{aligned} \tag{18}
$$

Then we have:

$$
\begin{aligned}
&\| y(t) - x(t) \| \\
&= \sum_{i=1}^{n} \sup_t \{e^{-Nt} \mid y_i(t) - x_i(t) \mid\} \\
&\le \sum_{i=1}^{n} \sup_{t\in[-\tau,0]} \{e^{-Nt} \mid \varphi_i(t) - \phi_i(t) \mid\} + \frac{c^*}{N^\alpha} \sum_{i=1}^{n} \sup_t \{e^{-Nt} \mid y_i(t) - x_i(t) \mid\} \\
&+ \| y(t) - x(t) \| \sum_{i=1}^{n} \frac{(a_i + b_i)L}{N^\alpha} + \| \varphi(t) - \phi(t) \| \sum_{i=1}^{n} \frac{b_i L}{N^\alpha} \\
&= (1 + \frac{\| B \| L}{N^\alpha}) \| \varphi(t) - \phi(t) \| + \frac{c^* + (\| A \| + \| B \|)L}{N^\alpha} \| y(t) - x(t) \| .
\end{aligned} \tag{19}
$$

It follows from (19) that:

$$
\| y(t) - x(t) \| \le \frac{1 + \frac{\|B\|L}{N^\alpha}}{1 - \frac{c^* + (\|A\| + \|B\|)L}{N^\alpha}} \| \varphi(t) - \phi(t) \| . \tag{20}
$$

Here we choose N large enough such that $(c^* + (\| A \| + \| B \|)L) / N^\alpha < 1$, then for $\forall \varepsilon > 0$, there exists $\delta = \frac{1 - \frac{c^* + (\|A\| + \|B\|)L}{N^\alpha}}{1 + \frac{\|B\|L}{N^\alpha}} \varepsilon > 0$ such that $\| x(t) - y(t) \| < \varepsilon$ when $\| \varphi(t) - \phi(t) \| < \delta$, which proves the solution of system (6) is uniformly stable.

Remark 1. It should be noted that when activation functions at time t and $t-\tau$ have the same form, the obtained results in this paper improve and extend the work presented in [18].

Remark 2. To the best of our knowledge, whatever in the area of theoretical research or numerical simulations, related results on the stability analysis of fractional-order neural networks with variable coefficients have not yet seen.

4. Analysis of The Proposed Results

An illustrative example is given to compare the main results studied in this paper with results proposed in [18].

Consider a class of fractional-order delayed neural networks described by the following differential equation:

$$\begin{cases} D^{\alpha}x_1(t) = -2x_1(t) + 0.75 f_1(x_1(t)) - 0.4 f_2(x_2(t)) - 0.15 f_1(x_1(t-\tau)) \\ \qquad\qquad + 0.1 f_2(x_2(t-\tau)) - 1.7, \\ D^{\alpha}x_2(t) = -x_2(t) - 0.25 f_1(x_1(t)) + 0.6 f_2(x_2(t)) - 0.12 f_1(x_1(t-\tau)) \\ \qquad\qquad - 0.7 f_2(x_2(t-\tau)) + 1.2, \end{cases} \tag{21}$$

Where the fractional order α is chosen as $\alpha = 0.7$, the activation functions are described by $f_1(x) = f_2(x) = |x+0.6| - |x-0.5|$, and the time delay $\tau = 0.01$.

Obviously, in system (21), $\|C\| = \max\{c_1, c_2\} = \max\{2,1\} = 2, \|A\| = 1.35, \|B\| = 0.85,\ L = 2$. Under the above parameters, we have $\|A\|L + \|B\|L > \min\{1-c_1, c_2\}$, hence the assumption 2 made in [18] is not satisfied. However, system (21) has a unique uniformly stable solution according to Theorem 1 and Theorem 2.

In fact, system (21) has a unique fixed point, which satisfies:

$$\begin{cases} -2x_1^* + 0.6 f_1(x_1^*) - 0.3 f_2(x_2^*) - 1.7 = 0, \\ -x_2^* - 0.37 f_1(x_1^*) - 0.1 f_2(x_2^*) + 1.2 = 0. \end{cases} \tag{22}$$

By virtue of Matlab, we can compute that the fixed point is $x^* = (-1.345, 1.497)^T$. Figure 1 shows that the solution of system (21) converges to the fixed point x^* in the time domain.

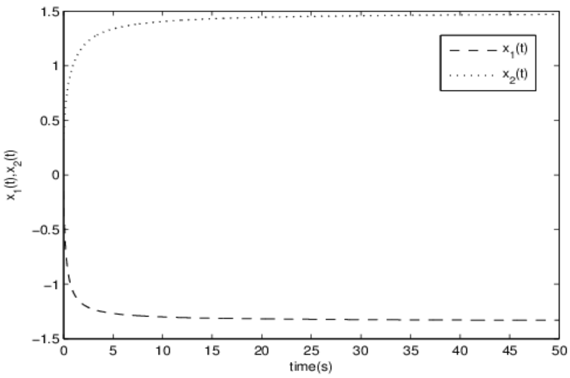

Figure 1. The dynamic behavior of system (23)

5. Conclusion

In this paper, the uniform stability problem is discussed for a class of fractional-order neural networks with variable coefficients and multiple time delays, a criteria on the existence and uniqueness of solutions and the uniform stability of solutions is established for this kind of neural networks. Finally, an example is given to demonstrate the effectiveness of our results.

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