

NOTICE: THIS MATERIAL MAY BE
PROTECTED BY COPYRIGHT LAW
(TITLE 17 U.S. CODE)

Absolute Stability of Global Pattern Formation and Parallel Memory Storage by Competitive Neural Networks

MICHAEL A. COHEN AND STEPHEN GROSSBERG

Abstract—The process whereby input patterns are transformed and stored by competitive cellular networks is considered. This process arises in such diverse subjects as the short-term storage of visual or language patterns by neural networks, pattern formation due to the firing of morphogenetic gradients in developmental biology, control of choice behavior during macromolecular evolution, and the design of stable context-sensitive parallel processors. In addition to systems capable of approaching one of perhaps infinitely many equilibrium points in response to arbitrary input patterns and initial data, one finds in these subjects a wide variety of other behaviors, notably traveling waves, standing waves, resonance, and chaos. The question of what general dynamical constraints cause global approach to equilibria rather than large amplitude waves is therefore of considerable interest. In another terminology, this is the question of whether global pattern formation occurs. A related question is whether the global pattern formation property persists when system parameters slowly change in an unpredictable fashion due to self-organization (development, learning). This is the question of absolute stability of global pattern formation. It is shown that many model systems which exhibit the absolute stability property can be written in the form

$$\frac{dx_i}{dt} = a_i(x_i) \left[b_i(x_i) - \sum_{k=1}^n c_{ik} d_k(x_k) \right] \quad (1)$$

$i = 1, 2, \dots, n$, where the matrix $C = \|c_{ik}\|$ is symmetric and the system as a whole is competitive. Under these circumstances, this system defines a

Manuscript received August 1, 1982; revised April 4, 1983. This work was supported in part by the National Science Foundation under Grant NSF IST-80-00257 and in part by the Air Force Office of Scientific Research under Grant AFOSR 82-0148.

The authors are with the Center for Adaptive Systems, Department of Mathematics, Boston University, Boston, MA 02215.

global Liapunov function. The absolute stability of systems with infinite but totally disconnected sets of equilibrium points can then be studied using the LaSalle invariance principle, the theory of several complex variables, and Sard's theorem. The symmetry of matrix C is important since competitive systems of the form (1) exist wherein C is arbitrarily close to a symmetric matrix but almost all trajectories persistently oscillate, as in the voting paradox. Slowing down the competitive feedback without violating symmetry, as in the systems

$$\frac{dx_i}{dt} = a_i(x_i) \left[b_i(x_i) - \sum_{k=1}^n c_{ik} d_k(y_k) \right]$$

$$\frac{dy_i}{dt} = e_i(x_i) [f_i(x_i) - y_i],$$

also enables sustained oscillations to occur. Our results thus show that the use of fast symmetric competitive feedback is a robust design constraint for guaranteeing absolute stability of global pattern formation.

I. INTRODUCTION: ABSOLUTE STABILITY OF GLOBAL PATTERN FORMATION IN SELF-ORGANIZING NETWORKS

THIS ARTICLE proves a global limit theorem for a class of n -dimensional competitive dynamical systems that can be written in the form

$$\dot{x}_i = a_i(x_i) \left[b_i(x_i) - \sum_{k=1}^n c_{ik} d_k(x_k) \right], \quad (1)$$

$i = 1, 2, \dots, n$, where the coefficients $\|c_{ij}\|$ form a symmetric

ric matrix. The systems (1) are more general in some respects but less general in other respects than the *adaptation level* competitive dynamical systems

$$\dot{x}_i = a_i(x) [b_i(x_i) - c(x)] \quad (2)$$

where $x = (x_1, x_2, \dots, x_n)$ and $i = 1, 2, \dots, n$, that have previously been globally analyzed (Grossberg [14], [18], [21]). To clarify the significance of the present theorem, some of the varied physical examples that can be written in the form (1) are summarized in this section. Section II indicates how these examples physically differ from related examples wherein sustained oscillations of various types can occur. Section III begins the mathematical development of the article.

System (1) includes the nonlinear neural networks

$$\dot{x}_i = -A_i x_i + (B_i - C_i x_i) [I_i + f_i(x_i)] - (D_i x_i + E_i) \left[J_i + \sum_{k=1}^n F_{ik} g_k(x_k) \right], \quad (3)$$

$i = 1, 2, \dots, n$. In (3), x_i is the potential, or short-term memory activity, of the i th cell (population) v_i in the network. Term $-A_i x_i$ describes the passive decay of activity at rate $-A_i$. Term

$$(B_i - C_i x_i) [I_i + f_i(x_i)] \quad (4)$$

describes how an excitatory input I_i and an excitatory feedback signal $f_i(x_i)$ increase the activity x_i . If $C_i = 0$, then term (4) describes an additive effect of input and feedback signal on activity [10]. If $C_i > 0$, then the input and feedback signal become ineffective when $x_i = B_i C_i^{-1}$ since then $B_i - C_i x_i = 0$. In this case, term (4) describes a shunting or multiplicative effect of input and feedback signal on activity. In a shunting network, the initial value inequality $x_i(0) \leq B_i C_i^{-1}$ implies that $x_i(t) \leq B_i C_i^{-1}$ for all $t \geq 0$, as occurs in nerve cells which obey the membrane equation (Hodgkin [24], Katz [26], Kuffler and Nicholls [28]). Term

$$-(D_i x_i + E_i) \left[J_i + \sum_{k=1}^n F_{ik} g_k(x_k) \right] \quad (5)$$

in (3) describes how an inhibitory input J_i and inhibitory feedback signals $F_{ik} g_k(x_k)$ from cell v_k to v_i decrease the activity x_i of v_i . If $D_i = 0$, then (5) describes an additive effect of input and feedback signals on activity. If $D_i > 0$, then the input and feedback signals become ineffective when $x_i = -D_i^{-1} E_i$, since then $D_i x_i + E_i = 0$. In this case, (5) describes a shunting effect of input and feedback signals on activity. An initial value choice $x_i(0) \geq -D_i^{-1} E_i$ implies that $x_i(t) \geq -D_i^{-1} E_i$ for all $t \geq 0$. Thus in a shunting network, but not an additive network, each activity $x_i(t)$ is restricted to a finite interval for all time $t \geq 0$. Suitably designed shunting networks can automatically retune their sensitivity to maintain a sensitive response within these finite intervals even if their inputs fluctuate in size over a much broader dynamic range (Grossberg [12], [21]).

The networks (1) are part of a mathematical classification theory, reviewed in [21], which characterizes how

prescribed changes in system parameters alter the transformation from input patterns $(I_1, I_2, \dots, I_n, J_1, J_2, \dots, J_n)$ into activity patterns (x_1, x_2, \dots, x_n) . In addition to the study of prescribed transformations, the mathematical classification theory seeks the most general classes of networks wherein important general processing requirements are guaranteed. In the present article, we study a class of networks which transform arbitrary input patterns into activity patterns that are then stored in short-term memory until a future perturbation resets the stored pattern. This property, also called *global pattern formation*, means that given any physically admissible input pattern $(I_1, I_2, \dots, I_n, J_1, J_2, \dots, J_n)$ and initial activity pattern $x(0) = (x_1(0), x_2(0), \dots, x_n(0))$, the limit $x(\infty) = \lim_{t \rightarrow \infty} (x_1(t), x_2(t), \dots, x_n(t))$ exists. The networks (1) include examples wherein nondenumerably many equilibrium points $x(\infty)$ exist (Grossberg [12], [21]).

A related property is the *absolute stability* of global pattern formation, which means that global pattern formation occurs given *any* choice of parameters in (1). The absolute stability property is of fundamental importance when (1) is part of a self-organizing (e.g., developing, learning) system, as in [15], [19]. Then network parameters can slowly change due to self-organization in an unpredictable way. Each new parameter choice may determine a different transformation from input pattern to activity pattern. An absolute stability theorem guarantees that, whatever transformation occurs, the network's ability to store the activity pattern is left invariant by self-organization. Thus the identification of an absolutely stable class of systems constrains the mechanisms of self-organization with which a system can interact without becoming destabilized in certain input environments.

The neural networks (3) include a number of models from population biology, neurobiology, and evolutionary theory. The Volterra-Lotka equations

$$\dot{x}_i = G_i x_i \left(1 - \sum_{k=1}^n H_{ik} x_k \right) \quad (6)$$

of population biology are obtained when $A_i = C_i = I_i = E_i = J_i = 0$ and $f_i(w) = g_i(w) = w$ for all $i = 1, 2, \dots, n$. The related Gilpin and Ayala system [6]

$$\dot{x}_i = G_i x_i \left[1 - \left(\frac{x_i}{K_i} \right)^{\theta_i} - \sum_{k=1}^n H_{ik} \left(\frac{x_k}{K_k} \right) \right] \quad (7)$$

is obtained when $A_i = C_i = I_i = E_i = J_i = 0$, $f_i(w) = 1 - w^{\theta_i} K_i^{-\theta_i}$ and $g_i(w) = w K_i^{-1}$ for all $i = 1, 2, \dots, n$.

The Hartline-Ratliff equation [34]

$$r_i = e_i - \sum_{k=1}^n K_{ik} \max(r_k - r_{ik}^{(0)}, 0) \quad (8)$$

for the steady-state outputs r_i of the Limulus retina arises as the equation of equilibrium of an additive network ($C_i = D_i = 0$) if, in addition, $f_i(w) = 0$ and $g_i(w) = \max(w - L_i, 0)$ for all $i = 1, 2, \dots, n$ (Grossberg [8], [9]).

The Eigen and Schuster equation [4]

$$\dot{x}_i = x_i \left(m_i x_i^{p-1} - q \sum_{k=1}^n m_k x_k^p \right) \quad (9)$$

for the evolutionary selection of macromolecular quasi-species is a special case of (3) such that $A_i = C_i = I_i = E_i = J_i = 0$, $B_i = F_{ik} = 1$, $D_i = q$, and $f_i(w) = g_i(w) = m_i x_i^p$ for all $i, k = 1, 2, \dots, n$. Feedback interactions among excitatory and inhibitory morphogenetic substances leading to "firing," or contrast enhancement, of a morphogenetic gradient can also be modeled by shunting networks (Grossberg [13], [16], [20]).

II. SOME SOURCES OF SUSTAINED OSCILLATIONS

The tendency of the trajectories of (1) to approach equilibrium points is dependent on the symmetry of the matrix $\|c_{ij}\|$ of interaction coefficients. Examples exist wherein the coefficient matrix may be chosen as close to a symmetric matrix as one pleases, yet almost all trajectories persistently oscillate even if all the functions $a_i(x_i)$, $b_i(x_i)$, and $d_k(x_k)$ are linear functions of their arguments. The May and Leonard model [33] of the voting paradox is illustrative. This model is defined by the three-dimensional system

$$\begin{aligned}\dot{x}_1 &= x_1(1 - x_1 - \alpha x_2 - \beta x_3) \\ \dot{x}_2 &= x_2(1 - \beta x_1 - x_2 - \alpha x_3) \\ \dot{x}_3 &= x_3(1 - \alpha x_1 - \beta x_2 - x_3).\end{aligned}\quad (10)$$

Grossberg [17] and Schuster *et al.* [36] proved that if $\beta > 1 > \alpha$ and $\alpha + \beta > 2$, then all positive trajectories except the uniform trajectories $x_1(0) = x_2(0) = x_3(0)$ persistently oscillate as $t \rightarrow \infty$. The matrix

$$\begin{pmatrix} 1 & \alpha & \beta \\ \beta & 1 & \alpha \\ \alpha & \beta & 1 \end{pmatrix} \quad (11)$$

can be chosen arbitrarily close to a symmetric matrix by letting α and β approach one without violating the hypotheses of Grossberg's theorem.

In a neural network such as (3), the hypothesis that the coefficient matrix $\|F_{ij}\|$ is symmetric is justified when the inhibitory interaction strengths F_{ij} and F_{ji} between cell v_i and cell v_j depend on the intercellular distance. Thus the tendency of the trajectories of (1) to approach equilibrium is interpreted in physical examples as a consequence of intercellular geometry.

The tendency to approach equilibrium also depends upon the rapidity with which feedback signals are registered. In (3), for example, the excitatory and inhibitory feedback signals $f_i(x_i)$ and $F_{ik}g_k(x_k)$, respectively, both depend explicitly on the excitatory activities x_i . *In vivo* these feedback signals are often emitted by interneuronal cells that are activated by the activities x_i before they return signals to v_i . Then (3) is replaced by the more general system

$$\begin{aligned}\dot{x}_i &= -A_i x_i + (B_i - C_i x_i) [I_i + f_i(w_i)] \\ &\quad - (D_i x_i + E_i) \left[J_i + \sum_{k=1}^n F_{ik} g_k(y_k) \right]\end{aligned}\quad (12)$$

$$\dot{w}_i = U_i(x_i) [W_i(x_i) - w_i] \quad (13)$$

$$\dot{y}_i = V_i(x_i) [Y_i(x_i) - y_i] \quad (14)$$

where w_i is the potential of an excitatory interneuron and y_i is the potential of an inhibitory interneuron that is activated by x_i . Large amplitude standing and traveling periodic waves have been found in continuum analogs of (12)–(14) (Ellias and Grossberg [5]). System (12)–(14) is more general than (3) because (12)–(14) reduce to a system of the form (3) when both w_i and y_i equilibrate very rapidly to fluctuations in x_i . Thus the tendency to approach equilibrium in (1) is due to both the symmetry and the speed of its feedback signals. Often as one perturbs off system (3) to a system of the form (12)–(14), one finds limiting patterns followed by standing waves followed by traveling waves [5]. In the neural network theory of short-term memory storage, both limiting patterns and standing waves are acceptable storage mechanisms; see [15], [19] for physical background. One approach to achieving these properties is to prove directly the global existence of limiting patterns for fast feedback systems such as (1), as we do in this article, and then to perturb off (1) by slowing down the feedback to characterize the parameter region wherein large amplitude standing waves are found before they bifurcate into large amplitude traveling waves.

Much more complex oscillations can also be inferred to exist in neural networks due to a mathematical relationship that exists between neural networks and models of individual nerve cells wherein complex oscillations have been proved to exist (Carpenter [1], [2]). This relationship allows the inference that traveling bursts and chaotic waveforms can be generated by suitably designed networks. To see why this is so, consider the following generalization of system (12)–(14):

$$\begin{aligned}\dot{x}_i &= -A_i x_i + (B - C_i x_i) \left[I_i + \sum_{k=1}^n f_{ik}(w_k) z_{ik} \right] \\ &\quad - (D_i x_i + E_i) \left[J_i + \sum_{k=1}^n g_{ik}(y_k) \right]\end{aligned}\quad (15)$$

$$\dot{w}_i = U_i(x_i) [W_i(x_i) - w_i] \quad (13)$$

$$\dot{y}_i = V_i(x_i) [Y_i(x_i) - y_i] \quad (14)$$

and

$$\dot{z}_{ik} = M_{ik} - N_{ik} z_{ik} - P_{ik} f_{ik}(w_k) z_{ik}. \quad (16)$$

Equation (15) permits excitatory feedback signaling from a cell v_k to v_i via the term $f_{ik}(w_k) z_{ik}$, as well as inhibitory feedback signaling via the term $g_{ik}(y_k)$. The new terms z_{ik} gate the excitatory feedback signal $f_{ik}(w_k)$ before it reaches v_i . *In vivo* such a gating action often corresponds to the release of a chemical transmitter at a rate proportional to $f_{ik}(w_k) z_{ik}$. Correspondingly, term $M_{ik} - N_{ik} z_{ik}$ in (16) describes the transmitter's slow accumulation to an asymptote $M_{ik} N_{ik}^{-1}$, whereas term $P_{ik} f_{ik}(w_k) z_{ik}$ describes the removal of transmitter at a rate proportional to $f_{ik}(w_k) z_{ik}$ (Grossberg [8], [11]). Equation (16) can be rewritten, analogous to (13) and (14), in the form

$$\dot{z}_{ik} = Q_{ik}(w_k) [Z_{ik}(w_k) - z_{ik}]. \quad (17)$$

However, whereas $W_i(x_i)$ and $Y_i(x_i)$ in (13) and (14) are

increasing functions of x_i ,

$$Z_{ik}(w_k) = M_{ik} [N_{ik} + P_{ik} f_{ik}(w_k)]^{-1} \quad (18)$$

is a decreasing function of w_k . Often *in vivo* the excitatory interneuronal potential w_i equilibrates rapidly to x_i in (13). Then $Z_{ik}(w_k)$ may be approximated by a decreasing function of x_i . When this is true, the variables w_i , y_i , and z_{ik} play a role in the network that is formally analogous to the role played by the variables m , n , and h of the Hodgkin-Huxley equations for nerve impulse transmissions [1], [2]. By relabeling cells appropriately, letting w_i rapidly equilibrate to x_i , and making a special choice of parameters and signals, the sum $-A_i x_i + (B - C_i x_i) \sum_{k=1}^n f_{ik}(w_k) z_{ik}$ in (15) can be rewritten in the form

$$D(x_{i-1} + x_{i+1} - 2x_i) + (B - x_i) h_i(x_i) z_i. \quad (19)$$

Term $D(x_{i-1} + x_{i+1} - 2x_i)$ plays the role of the diffusion term in the Hodgkin-Huxley equations. Carpenter's results on bursts and chaotic waves therefore hold in neural networks just so long as a spatially discrete version of the Hodgkin-Huxley equations can also support these waves.

Our concern in this article is not, however, to generate complex traveling waves but rather to rule them out. To accomplish this in a robust fashion, we turn to (1) because it eliminates both the waves due to fast feedback in an asymmetric geometry and the waves due to slow feedback in a symmetric geometry.

III. A GLOBAL LIAPUNOV FUNCTION

The adaptation level competitive systems

$$\dot{x}_i = a_i(x) (b_i(x_i) - c(x)) \quad (2)$$

were globally analyzed by associating a suitable Liapunov functional $M^+(x_i)$ to every such system. This functional, which is an integral of a maximum function

$$M^+(x_i) = \int_0^1 \max_v [b_i(x_i(v)) - c(x(v))] dv, \quad (20)$$

permitted a concept of *jump*, or *decision*, to be associated with (2). Using this concept, the idea could be explicated that the decision schemes of adaptation level systems are globally consistent and thereby cause every trajectory to approach an equilibrium point [14], [18]. By contrast, when the same method was applied to the voting paradox system (10), it was found that the decision scheme of this system is globally inconsistent, and thus almost all trajectories persistently oscillate [17], [18]. Although every competitive system defines such a Liapunov functional and a decision scheme, this method has not yet succeeded in proving that the decision scheme of (1) is globally consistent. Such a theorem is greatly to be desired.

In its absence, we have found that the systems (1) admit a global Liapunov function which can be analyzed. A considerable amount of work has already been done on finding Liapunov functions for special cases of (1). For example, a Liapunov function which proves local asymptotic stability of isolated equilibrium points of Volterra-Lotka systems was described in a classical paper

of MacArthur [32]. Global Liapunov functions for Volterra-Lotka and Gilpin-Ayala systems have been found in cases where only one equilibrium point exists (Goh and Agnew [7]). This constraint is much too strong in systems that are designed to transform and store a large variety of patterns. Our analysis includes systems which possess infinitely many equilibrium points. Liapunov functions have also been described for Volterra-Lotka systems whose off-diagonal interaction terms are relatively small (Kilmer [27], Takeuchi *et al.* [37]). We do not need this type of constraint to derive our results.

The function

$$V(x) = - \sum_{i=1}^n \int_0^{x_i} b_i(\xi_i) d_i'(\xi_i) d\xi_i + \frac{1}{2} \sum_{j,k=1}^n c_{jk} d_j(x_j) d_k(x_k) \quad (21)$$

is a global Liapunov function for (1) because

$$\dot{V}(x) = - \sum_{i=1}^n a_i(x_i) d_i'(x_i) \left[b_i(x_i) - \sum_{k=1}^n c_{ik} d_k(x_k) \right]^2 \quad (22)$$

Function $\dot{V}(x) \leq 0$ along trajectories just so long as every function $d_i(x_i)$ is monotone nondecreasing. This condition implies that (1) is competitive. In (3), where $d_i \equiv g_i$, the condition means that inhibitory feedback $g_i(x_i)$ cannot decrease as activity x_i increases. Systems (1) can, in fact, be written in the gradient form

$$\dot{x} = A(x) \nabla B(x) \quad (23)$$

if each function $d_i(x_i)$ is strictly increasing by choosing the matrix $A(x) = \|A_{ij}(x)\|$ to satisfy

$$A_{ij}(x) = \frac{a_i(x_i) \delta_{ij}}{d_i'(x_i)} \quad (24)$$

and $B(x) = -V(x)$.

The standard theorems about Liapunov functions and gradient representations imply that each trajectory converges to the largest invariant set M contained in the set E where [22]

$$\frac{d}{dt} V = 0. \quad (25)$$

Given definition (21) of $V(x)$, it is easy to see that points in E are equilibrium points if each function $d_i(x_i)$ is strictly increasing. It still remains to show in this case that each trajectory approaches a unique equilibrium point, although for all practical purposes every trajectory that approaches M becomes approximately constant in any bounded interval of sufficiently large times.

Further argument is required when each function $d_i(x_i)$ is not strictly increasing, which is the typical situation in a neural network. There each inhibitory feedback signal function $d_i(x_i)$ can possess an *inhibitory signal threshold* Γ_i^- such that $d_i(x_i) = 0$ if $x_i \leq \Gamma_i^-$ and $d_i'(x_i) > 0$ if $x_i > \Gamma_i^-$. Since each $d_i(x_i)$ is still monotone nondecreasing, although not strictly increasing, function $V(x)$ in (21)

continued to define a Liapunov function. Consequently, every trajectory still converges to the invariant set M . However, further analysis is now required to guarantee that M consists of equilibrium points, let alone isolated equilibrium points. Even in the cases wherein no such degeneracy occurs, it has not previously been noticed that so many physically important examples can be written in the form (1) and that (1) admits a global Liapunov function.

IV. APPLICATION OF THE LASALLE INVARIANCE PRINCIPLE

We will study the general system

$$\dot{x}_i = a_i(x_i) \left[b_i(x_i) - \sum_{k=1}^n c_{ik} d_k(x_k) \right] \quad (1)$$

under hypotheses that include the shunting competitive neural networks

$$\begin{aligned} \dot{y}_i = & -A_i y_i + (B_i - C_i y_i) [I_i + f_i(y_i)] \\ & - (D_i y_i + E_i) \left[J_i + \sum_{k=1}^n F_{ik} g_k(y_k) \right]. \end{aligned} \quad (26)$$

In the shunting case, $C_i \neq 0 \neq D_i$. The simpler additive neural networks wherein $C_i = 0 = D_i$ are also included in our analysis but will not be explicitly discussed. In the shunting case, (26) can be rewritten without loss of generality in the form

$$\begin{aligned} \dot{y}_i = & -A_i y_i + (B_i - y_i) [I_i + f_i(y_i)] \\ & - (y_i + C_i) \left[J_i + \sum_{k=1}^n F_{ik} g_k(y_k) \right] \end{aligned} \quad (27)$$

by a suitable redefinition of terms.

We distinguish x_i in (1) from y_i in (27) because our hypotheses hold when

$$x_i = y_i + C_i. \quad (28)$$

Then (27) reduces to (1) via the definitions

$$a_i(x_i) = x_i, \quad (29)$$

$$b_i(x_i) = x_i^{-1} \{ A_i C_i - (A_i + J_i) x_i + (B_i + C_i - x_i) [I_i + f_i(x_i - C_i)] \}, \quad (30)$$

$$c_{ik} = F_{ik}, \quad (31)$$

and

$$d_k(x_k) = g_k(x_k - C_k). \quad (32)$$

Our first task is to prove that $V(x)$ is a Liapunov function of x in the positive orthant \mathbb{R}_+^n . To do this, we study (1) under the following hypotheses:

- symmetry:** matrix $\|c_{ij}\|$ is a symmetric matrix of non-negative constants;
- continuity:** function $a_i(\xi)$ is continuous for $\xi \geq 0$; function $b_i(\xi)$ is continuous for $\xi > 0$;
- positivity:** function $a_i(\xi) > 0$ for $\xi > 0$; function $d_i(\xi) \geq 0$ for $\xi \in (-\infty, \infty)$.

d) **smoothness and monotonicity:** function $d_i(\xi)$ is differentiable and monotone nondecreasing for $\xi \geq 0$.

To prove that $V(x)$ is a Liapunov function, we first show that positive initial data generate positive bounded trajectories of (1), henceforth called *admissible* trajectories. This can be shown if two more hypotheses are assumed. The choice of hypotheses (34)–(36) below is influenced by the fact that function b_i in (30) may become unbounded as $x_i \rightarrow 0^+$.

Lemma 1 (Boundedness and Positivity):

Boundedness: For each $i = 1, 2, \dots, n$, suppose that

$$\limsup_{\xi \rightarrow \infty} [b_i(\xi) - c_{ii} d_i(\xi)] < 0. \quad (33)$$

Positivity: For each $i = 1, 2, \dots, n$, suppose either that

$$\lim_{\xi \rightarrow 0^+} b_i(\xi) = \infty \quad (34)$$

or that

$$\lim_{\xi \rightarrow 0^+} b_i(\xi) < \infty \quad (35)$$

and

$$\int_0^\epsilon \frac{d\xi}{a_i(\xi)} = \infty \quad \text{for some } \epsilon > 0. \quad (36)$$

Then any positive initial data generate an admissible trajectory.

Proof: Boundedness is proved using (33) as follows. Inequality

$$b_i(x_i) - \sum_{k=1}^n c_{ik} d_k(x_k) \leq b_i(x_i) - c_{ii} d_i(x_i) \quad (37)$$

is true because all c_{ik} and d_k are nonnegative. Since also $a_i(x_i)$ is positive at large x_i values, (37) shows that $(d/dt)x_i < 0$ at large x_i values. Indeed, given any positive initial data, an $L_i < \infty$ exists such that $x_i(t) \leq L_i$ at sufficiently large times t , $i = 1, 2, \dots, n$.

Condition (34) implies positivity because each term $\sum_{k=1}^n c_{ik} d_k(x_k)$ is bounded if all $x_k \leq L_k$, $k = 1, 2, \dots, n$; hence term $b_i(x_i) - \sum_{k=1}^n c_{ik} d_k(x_k)$ becomes positive if all $x_k \leq L_k$, $k = 1, 2, \dots, n$ as $x_i \rightarrow 0^+$. Since also $a_i(x_i) > 0$ for $x_i > 0$, $(d/dt)x_i > 0$ before x_i reaches 0, hence x_i can never reach zero.

If (35) and (36) hold, then at the first time $t = T$ such that $x_i(T) = 0$,

$$\begin{aligned} -\infty &= \int_{x_i(0)}^0 \frac{d\xi}{a_i(\xi)} \\ &= \int_0^T \left[b_i(x_i(t)) - \sum_{k=1}^n c_{ik} d_k(x_k(t)) \right] dt > -\infty. \end{aligned} \quad (38)$$

which is a contradiction. Hence $x_i(t)$ remains positive for all $t \geq 0$.

Using the fact that positive initial data generate admissible trajectories, we can easily verify that the function

$$V(x) = - \sum_{k=1}^n \int_0^{x_k} b_i(\xi_i) d_i'(\xi_i) d\xi_i + \frac{1}{2} \sum_{j,k=1}^n c_{jk} d_j(x_j) d_k(x_k) \quad (21)$$

is a Liapunov function.

Proposition 1 (Liapunov Function): The function $V(x)$ satisfies

$$\frac{d}{dt} V(x(t)) \leq 0 \quad (39)$$

on admissible trajectories.

Proof: By direct computation,

$$\frac{d}{dt} V(x(t)) = - \sum_{i=1}^n a_i(x_i(t)) d_i'(x_i(t)) \left[b_i(x_i(t)) - \sum_{k=1}^n c_{ik} d_k(x_k(t)) \right]^2 \quad (22)$$

Since $a_i \geq 0$ on admissible trajectories and $d_i' \geq 0$ by hypothesis, (39) follows.

In some cases where d_i admits a threshold, d_i' is only piecewise differentiable. In these cases, the trajectory derivative $(d/dt)V$ can be replaced by

$$D^+ V(x) = \liminf_{h \rightarrow 0^+} \frac{1}{h} [V(x + h\dot{x}) - V(x)] \quad (40)$$

and the Riemann integral $\int_0^{x_i} b_i(\xi_i) d_i'(\xi_i) d\xi_i$ in the definition of $V(x)$ can be replaced by a Radon integral.

To apply the LaSalle invariance principle [22], [29], [30] to $V(x)$, we also need to guarantee that $V(x)$ is bounded and continuous on admissible trajectories.

Proposition 2: If the hypotheses of Lemma 1 hold, then $V(x)$ (or a simple redefinition thereof) is bounded and continuous on admissible trajectories.

Proof: If (35) holds, then the integrals

$$\int_0^{x_i} b_i(\xi_i) d_i'(\xi_i) d\xi_i \quad (41)$$

in (21) are bounded because admissible trajectories are bounded. The remaining terms

$$\sum_{j,k=1}^n c_{jk} d_j(x_j) d_k(x_k) \quad (42)$$

of (21) are bounded because the functions $d_j(x_j)$ are continuous functions of bounded variables.

If (34) holds but

$$\lim_{\xi \rightarrow 0^+} |b_i(\xi) d_i'(\xi)| < \infty, \quad (43)$$

then the same argument as above is valid. If (43) does not hold, then the integral $\int_0^{x_i}$ in (21) can be replaced by an integral $\int_{\lambda_i}^{x_i}$, where λ_i is a positive constant that is chosen below. Such a choice is possible due to several facts working together. Each d_k is a nonnegative and monotone

nondecreasing function of the variable x_k , where $0 \leq x_k \leq L_k$ at sufficiently large times, $k = 1, 2, \dots, n$. Consequently, a positive finite L exists such that

$$\sum_{k=1}^n c_{ik} d_k(x_k) \leq L \quad (44)$$

on all admissible trajectories at sufficiently large times. Since (34) holds, an interval $[0, 2\lambda_i]$ exists such that

$$b_i(x_i) - \sum_{k=1}^n c_{ik} d_k(x_k) \geq L \quad (45)$$

and thus

$$\dot{x}_i \geq La_i(x_i) \quad (46)$$

whenever $0 < x_i \leq 2\lambda_i$ on any admissible trajectory at sufficiently large times. Since function a_i is positive on any interval $[x_i(T), 2\lambda_i]$ where $x_i(T) > 0$, a_i has a positive lower bound on this interval. Thus by (46), if T is chosen so large that (44) holds for $t \geq T$, then $x_i(t)$ increases at least at a linear rate until it exceeds λ_i and remains larger than λ_i thereafter. Since this argument holds for any admissible trajectory, the choice of λ_i in the integral $\int_{\lambda_i}^{x_i}$ is justified.

Continuity follows by inspection of each term in (21), replacing the integral $\int_0^{x_i}$ by $\int_{\lambda_i}^{x_i}$ where necessary.

The LaSalle invariance principle therefore implies the following theorem.

Theorem 1 (Convergence of Trajectories): In any system

$$\dot{x}_i = a_i(x_i) \left[b_i(x_i) - \sum_{k=1}^n c_{ik} d_k(x_k) \right] \quad (1)$$

such that

- matrix $\|c_{ij}\|$ is symmetric and all $c_{ij} \geq 0$;
- function a_i is continuous for $\xi \geq 0$; function b_i is continuous for $\xi > 0$;
- function $a_i > 0$ for $\xi > 0$; function $d_i \geq 0$ for all ξ ;
- function d_i is differentiable and monotone nondecreasing for $\xi \geq 0$;
- $\limsup_{\xi \rightarrow \infty} [b_i(\xi) - c_{ii} d_i(\xi)] < 0$ for all $i = 1, 2, \dots, n$;
- and either

$$\lim_{\xi \rightarrow 0^+} b_i(\xi) = \infty \quad (34)$$

or

$$\lim_{\xi \rightarrow 0^+} b_i(\xi) < \infty \quad (35)$$

and

$$\int_0^\epsilon \frac{d\xi}{a_i(\xi)} = \infty \quad \text{for some } \epsilon > 0; \quad (36)$$

all admissible trajectories approach the largest invariant set M contained in the set

$$E = \left\{ y \in \mathbb{R}^n : \frac{d}{dt} V(y) = 0, y \geq 0 \right\}. \quad (47)$$

where

$$\frac{d}{dt} V = - \sum_{i=1}^n a_i d_i' \left[b_i - \sum_{k=1}^n c_{ik} d_k \right]^2. \quad (22)$$

Corollary 1: If each function d_i is strictly increasing, then the set E consists of equilibrium points of (1).

Proof: Because each function a_i and d_i' is nonnegative on admissible trajectories, each summand in (22) is nonnegative. Hence the result follows by inspection of (47) and (22).

V. DECOMPOSITION OF EQUILIBRIA INTO SUPRATHRESHOLD AND SUBTHRESHOLD VARIABLES

Our strategy for analyzing M when the functions d_i can have thresholds is to decompose the variables x_i into suprathreshold and subthreshold variables, and then to show how sets of suprathreshold equilibria can be used to characterize the ω -limit set of the full system (1). To say this more precisely, we now define some concepts.

The *inhibitory threshold* of d_i is a constant $\Gamma_i^- \geq 0$ such that

$$\left. \begin{aligned} d_i(\xi) &= 0, & \text{if } \xi \leq \Gamma_i^- \\ d_i'(\xi) &> 0, & \text{if } \xi > \Gamma_i^- \end{aligned} \right\}. \quad (48)$$

The function $x_i(t)$ is *suprathreshold* at t if $x_i(t) > \Gamma_i^-$ and *subthreshold* at t if $x_i(t) \leq \Gamma_i^-$. At any time t , suprathreshold variables receive signals only from other suprathreshold variables.

Because only suprathreshold variables signal other suprathreshold variables, we can first restrict attention to all possible subsets of suprathreshold values that occur in the ω -limit points $\omega(\gamma)$ of each admissible trajectory γ . Using the fact that each function d_i is strictly increasing in the suprathreshold range, we will show that the suprathreshold subset corresponding to each ω -limit point defines an equilibrium point of the subsystem of (1) that is constructed by eliminating all the subthreshold variables of that ω -limit point. We will show that the set of all such subsystem suprathreshold equilibrium points is countable. We can then show that under a weak additional hypothesis, the ω -limit set of each trajectory is an equilibrium point, and that the set of equilibrium points is totally disconnected. First we make a generic statement about almost all systems (1), and then we study particular classes of neural networks (3) whose global pattern formation properties can be directly verified.

VI. ALMOST ALL SUPRATHRESHOLD EQUILIBRIUM SETS ARE COUNTABLE

In this section, we observe that, for almost all choices of the parameters c_{ik} in (1), Sard's theorem routinely implies that the set of suprathreshold equilibrium points is countable [23], [25]. A generic statement can also be made by varying functions a_i , b_i , and d_i within the class C^1 by combining the Sard theorem with Fubini's theorem. The Sard theorem is stated as Theorem 2 for completeness.

Let X be an open set in \mathbb{R}^m , P an open set in \mathbb{R}^k , and Z an open set in \mathbb{R}^n . Let $S: X \times P \rightarrow Z$ be a C^1 map. A point $z \in \mathbb{R}^n$ is said to be a *regular value* of S if $\text{rank } dS(\cdot, \cdot) = n$ whenever $S(x, p) = z$, where dS denotes the $n \times (m+k)$ Jacobian matrix of S .

Theorem 2 (Sard): Let z be a regular value of S . Then z is a regular value of $S(\cdot, p)$ for almost all $p \in P$ in the sense of Lebesgue measure.

Corollary 2: Let each a_i , b_i , and d_i be in $C^1(0, \infty)$. Let P denote the matrix of parameters $\|c_{ik}\|$. Then a measure zero subset $Q \subset P$ exists such that the suprathreshold equilibria of (1) corresponding to parameters $p \in P \setminus Q$ are countable.

Proof: To consider the equilibrium points of (1), we let $z = 0$ and define the vector function $S = (S_1, S_2, \dots, S_n)$ by

$$S_i(x) = a_i(x_i) \left[b_i(x_i) - \sum_{k=1}^n c_{ik} d_k(x_k) \right], \quad i = 1, 2, \dots, n. \quad (49)$$

Then the points for which $S = 0$ are the equilibrium points of (1).

To prove that $dS(\cdot, \cdot)$ has rank n at the suprathreshold equilibria $S = 0$, we prove the stronger statement that $dS(\cdot, \cdot)$ has rank n at all suprathreshold vectors x ; that is, at all $x_i > \Gamma_i^- \geq 0$, $i = 1, 2, \dots, n$. By (49)

$$\frac{\partial S_i}{\partial c_{ii}} = -a_i(x_i) d_i(x_i) \quad (50)$$

where, by the positivity of a_i when $x_i > 0$ and the inhibitory threshold condition (48), $a_i(x_i) d_i(x_i) > 0$ at any suprathreshold value of x_i . The corresponding n rows and columns of dS form a diagonal submatrix whose i th entry is given by (50). Matrix dS therefore has rank n at all suprathreshold vectors x .

The main condition of Sard's theorem is hereby satisfied by this matrix S . Thus a set Q of measure zero exists such that $dS(\cdot, p)$ has rank n for all $p \in P \setminus Q$. Now the inverse function theorem can be used at each $p \in P \setminus Q$ to show that the suprathreshold equilibrium points x of $S(x, p) = 0$ are isolated, hence countable.

VII. ALL ω -LIMIT POINTS ARE EQUILIBRIA

Theorem 3 (Global Pattern Formation) Let all the hypotheses of Theorem 1 hold. Also suppose that no level sets of the functions b_i contain an open interval and that the subsystem suprathreshold equilibrium vectors are countable. Then each admissible trajectory converges to an equilibrium point.

Proof: Consider the ω -limit set $\omega(\gamma)$ of a given admissible trajectory γ . Since Theorem 1 holds, each component x_i of $x \in \omega(\gamma)$ satisfies either

$$a_i(x_i) \left[b_i(x_i) - \sum_{k=1}^n c_{ik} d_k(x_k) \right] = 0 \quad (51)$$

$$d_i'(x_i) = 0. \quad (52)$$

In the former case, x_i is suprathreshold; in the latter case, subthreshold.

Using this decomposition, we can show that a unique vector of subsystem suprathreshold values exists corresponding to each $\omega(\gamma)$ in the following way. The set $\omega(\gamma)$ is connected. If two or more vectors of subsystem suprathreshold values existed, an uncountable set of subsystem suprathreshold vectors would exist in $\omega(\gamma)$. This basic fact can be seen by projecting $\omega(\gamma)$ onto a coordinate where the two hypothesized vectors differ. The image of $\omega(\gamma)$ on this coordinate is a connected set. This fact, together with the definition of a suprathreshold value, implies that a nontrivial interval of suprathreshold values exists in this image. The inverse image of this interval therefore contains a nondenumerable set of subsystem suprathreshold vectors, a conclusion that contradicts the hypothesis that the set of subsystem suprathreshold vectors is countable. Hence no more than one subsystem suprathreshold vector exists in each $\omega(\gamma)$.

Using this fact, we now show that the subthreshold values of each $\omega(\gamma)$ are uniquely determined. Let $U(\gamma)$ be the indices of the unique subsystem suprathreshold vector ($x_i^* : i \in U(\gamma)$) of $\omega(\gamma)$. For every $i \notin U(\gamma)$, (1) can be rewritten as

$$\dot{x}_i = a_i(x_i)[b_i(x_i) - e_i] + \epsilon(t) \quad (53)$$

where the constant e_i satisfies

$$e_i = \sum_{k \in U(\gamma)} c_{ik} d_k(x_k^*) \quad (54)$$

and

$$\lim_{t \rightarrow \infty} \epsilon(t) = 0 \quad (55)$$

because a_i is bounded on admissible trajectories. To complete the proof, we use the fact that the level sets of b_i do not contain an open interval to conclude that each x_i , $i \in U(\gamma)$, has a limit. Since also each x_i , $i \in U(\gamma)$, has a limit, it will follow that each $\omega(\gamma)$ is an equilibrium point.

The proof shows that the ω -limit set of the one-dimensional equation (53) is a point. Suppose not. Since (53) defines a one-dimensional system, the ω -limit set, being connected, is then a nontrivial closed interval V_i . By hypothesis, the function $b_i - e_i$ in (53) cannot vanish identically on any nontrivial subinterval of V_i . Since function $b_i - e_i$ is continuous, a subinterval $W_i \subset V_i$ and an $\epsilon > 0$ exist such that either $b_i(\xi) - e_i \geq \epsilon$ if $\xi \in W_i$ or $b_i(\xi) - e_i \leq -\epsilon$ if $\xi \in W_i$. In either case, x_i will be forced off interval W_i at all sufficiently large times by (55) and the fact that $a_i > 0$ except when $x_i = 0$. Hence no nontrivial interval W_i can be contained in the ω -limit set of (53). This ω -limit set is thus a point, and the proof is complete.

Corollary 3 (Almost Absolute Stability): Consider the class of systems (1) such that

- 1) hypotheses a)–f) of Theorem 1 hold;
- 2) each function a_i , b_i , and d_i is in $C^1(0, \infty)$;
- 3) none of the level sets of b_i contains an open interval.

Then for almost all choices of the parameters c_{ik} , global pattern formation occurs.

Proof: The proof follows directly from Corollary 2 and Theorem 3.

The hypotheses of Theorem 3 allow us to conclude that the set of all equilibrium points of (1) is a totally disconnected set. A *totally disconnected set* is a set whose largest connected subset is a point.

Instead of considering the solutions of $b_i(\xi) = e_i$, corresponding to the ω -limit set $\omega(\gamma)$ of individual trajectories, as we did to prove Theorem 3, in this proof we consider the set of solutions of $b_i(\xi) = e_i$ generated by arbitrary admissible trajectories.

Theorem 4 (Totally Disconnected Equilibrium Set): Suppose that each b_i is continuous, that no level set of b_i contains an open interval, and that the system suprathreshold equilibrium vectors are countable. Then the set of all equilibrium points of (1) is totally disconnected.

Proof: Each choice of subsystem suprathreshold vector defines a constant value of e_i in (54). For fixed e_i , the level set

$$\{\xi : b_i(\xi) - e_i = 0\} \quad (56)$$

is nowhere dense, since if (56) were dense on some interval, the continuity of b_i would imply that the level set (56) contains an open interval, which is impossible.

By hypothesis, only countably many choices of e_i exist for each $i = 1, 2, \dots, n$. Since each set (56) is nowhere dense, the set of all subthreshold equilibrium solutions of (53) is a countable union of nowhere dense sets and is therefore nowhere dense by the Baire category theorem. By hypothesis, the set of all subsystem suprathreshold equilibrium solutions of (1) is countable. The set of all x_i corresponding to the subsystem suprathreshold equilibrium solutions of (1) is therefore also countable. The union P_i of the nowhere dense subthreshold set and the countable suprathreshold set is totally disconnected. The product set $X_{i-1}^n P_i$ is also totally disconnected. Since the set of all equilibria of (1) is contained in $X_{i-1}^n P_i$, it is totally disconnected.

VIII. NEURAL NETWORKS WITH FINITELY MANY SUPRATHRESHOLD EQUILIBRIUM POINTS

To remove the "almost all" from results such as Corollary 3, we consider various special cases that are of physical interest, notably the shunting competitive networks (27) with polynomial or sigmoid feedback signal functions. We write the networks (27) using the change of variables

$$x_i = y_i + C_i \quad (28)$$

to make the results comparable to previous results about (1). Then (27) can be written as

$$\dot{x}_i = S_i(x), \quad i = 1, 2, \dots, n \quad (57)$$

such that

$$S_i(x) = \alpha_i + (\beta_i - x_i) F_i(x_i) - x_i \left(\gamma_i + \sum_{k=1}^n c_{ik} G_k(x_k) \right) \quad (58)$$

where

$$\alpha_i = a_i c_i + (b_i + c_i) I_i, \tag{59}$$

$$\beta_i = b_i + c_i, \tag{60}$$

$$\gamma_i = a_i + I_i + J_i, \tag{61}$$

$$F_i(x_i) = f_i(x_i - c_i), \tag{62}$$

and

$$G_i(x_i) = g_i(x_i - c_i). \tag{63}$$

One natural approach to proving that only finitely many suprathreshold equilibrium points exist is to apply a basic theorem from the theory of several complex variables [35]. The following results illustrate rather than exhaust the applications of this theorem to our systems.

The theorem in question concerns analytic subvarieties of a connected open set Ω of $\mathbb{C}^n = \{n\text{-tuples of complex variables}\}$. A set $V \subset \Omega$ is an *analytic subvariety* of Ω if every point $p \in \Omega$ has a neighborhood $N(p)$ such that

$$V \cap N(p) = \bigcap_{i=1}^r Z(h_i) \tag{64}$$

where $Z(h_i)$ is the set of zeros of the function h_i holomorphic in $N(p)$. Our applications derive from the following theorem.

Theorem 5: Every compact analytic subvariety of a connected open set Ω is a finite set of points.

A general strategy for applying Theorem 5 to neural networks can be stated as five steps.

1) Choose the signal function F_i and G_i in (62) and (63), respectively, to be real analytic on their suprathreshold intervals.

2) Extend the definitions of F_i and G_i to make them complex analytic inside a sufficiently large open disk. (It does not matter that the analytic extension of the signal function to the subthreshold interval no longer agrees with the original definition of the function.)

3) Extend S_i in (58) to be an analytic function $\Phi_i(z)$ in an open connected set $\Omega_i \subset \mathbb{C}^n$.

4) Show that the solutions to the system of equations

$$\phi_i(z) = 0, \quad i = 1, 2, \dots, n \tag{65}$$

are contained in a bounded open set P whose closure is contained in $\Omega = \bigcap_{i=1}^n \Omega_i$. Since the set of zeros is closed, the set of zeros is a compact analytic subvariety of Ω , hence finite.

5) Set all imaginary parts of these zeros equal to zero to prove that finitely many suprathreshold equilibria exist.

The method is illustrated by the following three theorems.

Theorem 6 (Polynomial Signals): Let each function $F_i(\xi)$ and $G_i(\xi)$ be a polynomial in the suprathreshold domain $\xi \geq \Gamma_i^-$, and suppose that $\deg F_i > \deg G_j$ whenever $c_{ij} > 0$, $i, j = 1, 2, \dots, n$. Then only finitely many suprathreshold equilibrium points of (1) exist.

Proof: Analytically continue the functions $S_i(x)$, $x_i \geq \Gamma_i^-$, $i = 1, 2, \dots, n$ to be polynomial functions $\tilde{S}_i(z)$ of n complex variables z . The zeros of system $\tilde{S}_i(z) = 0$, $i =$

$1, 2, \dots, n$, are thus an analytic subvariety W of \mathbb{C}^n . We show that W is bounded, hence compact. Then using Theorem 5 and the fact that $S_i(x) = \tilde{S}_i(z)$ when z is real and $x_i \geq \Gamma_i^-$, $i = 1, 2, \dots, n$, it follows that at most finitely many suprathreshold equilibria of (57) exist.

Boundedness is easily proved as follows. Choose any $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$. Let z_i be the component of maximal modulus in z ; that is, $|z_i| \geq |z_j|$, $j \neq i$. Consider the highest degree term of $\tilde{S}_i(z)$. This term corresponds to the highest degree term of the analytic continuation of term $x_i F_i(x_i)$ in $S_i(x)$. If $|z|$ is chosen sufficiently large, the degree condition on the signal functions along with the inequalities $|z_i| \geq |z_j|$, $j \neq i$, imply that the modulus of this highest degree term exceeds the sum of moduli of all other terms in $\tilde{S}_i(z)$. Consequently, $\tilde{S}_i(z) \neq 0$ if $|z| \gg 0$. In other words, no zero exists of the full system $\tilde{S}_i(z) = 0$, $i = 1, 2, \dots, n$, outside some bounded ball in \mathbb{C}^n , and the proof is complete.

Corollary 4 (Polynomial Absolute Stability): Let system (57) be given with a symmetric matrix $\|c_{ij}\|$ of nonnegative interaction coefficients and signal functions that are polynomial in their suprathreshold region such that $\deg F_i > \deg G_j$ for all $c_{ij} > 0$ and each G_j has nonnegative coefficients. Then global pattern formation is absolutely stable within this class of networks.

The proof consists in verifying that the hypotheses of Theorems 1, 3, and 6 are satisfied.

Theorem 6 demonstrates that suprathreshold polynomial signal functions for which the norm of excitatory feedback grows more quickly than the norm of inhibitory feedback lead to global pattern formation. Any smooth signal functions can be uniformly approximated within this class of polynomials, but that does not imply that (58) has countably many zeros using these signal functions. The next result considers sigmoid signal functions to illustrate how Theorem 5 can be applied to a nonpolynomial case of great physical interest (Grossberg [12], [21]). Sigmoid signal functions, unlike polynomials, approach finite asymptotes at large activity values. Absolute stability holds within a class of sigmoid functions wherein a trade-off exists between the rate of signal growth, the asymptote of signal growth, and the spatial breadth and size of inhibitory interaction strengths.

To illustrate the factors that control sigmoid signal behavior, we consider sigmoid signal functions such that if $x_i \geq \Gamma_i^-$,

$$F_i(x_i) = \frac{p_i(x_i - \Gamma_i^-)^{N_i}}{q^{N_i} + (x_i - \Gamma_i^-)^{N_i}} \tag{66}$$

and

$$G_i(x_i) = \frac{(x_i - \Gamma_i^-)^{M_i}}{r^{M_i} + (x_i - \Gamma_i^-)^{M_i}} \tag{67}$$

where M_i and N_i are positive integers, $i = 1, 2, \dots, n$. The asymptote of G_i is set equal to one without loss of generality because G_i multiplies a coefficient c_{ij} in all its ap-

pearances in (55), and the symmetry $c_{ij} = c_{ji}$ is not needed in the following estimate.

Theorem 7 (Sigmoid Signals): Suppose that the parameters in (66) and (67) are chosen to satisfy the following three conditions

- 1) $\epsilon > 0$ and $\delta > 1$ exist such that

$$\max(b_i + c_i - \Gamma_i^-, q_i) < \epsilon < \delta\epsilon < r_i, \quad i=1, 2, \dots, n. \quad (68)$$

- 2) The constants

$$s_i = \sum_{k=1}^n c_{ik} (\delta^{M_k} - 1)^{-1} \quad (69)$$

satisfy the inequalities

$$2s_i < p_i, \quad i = 1, 2, \dots, n. \quad (70)$$

- 3) The inequality

$$(p_i - 2s_i)q_i > 2|\alpha_i - \gamma_i \Gamma_i^-| + p_i |\beta_i - \Gamma_i^-| + 2s_i \Gamma_i^- \quad (71)$$

holds, $i = 1, 2, \dots, n$.

Then at most finitely many suprathreshold equilibrium points of (57) exist.

Remark: Inequality (68) says that the excitatory signal functions change faster-than-linearly at smaller activities than the inhibitory signal functions, and that the turning points q_i and r_i are uniformly separated across signal functions. Inequality (70) says that the excitatory feedback elicited by large activities dominates the total inhibitory feedback elicited by these activities. These two inequalities are thus analogous to the conditions on polynomial degrees in the previous theorem. The left-hand side of inequality (71) refines these constraints by requiring the faster-than-linear range of the excitatory signal function to occur at large activities if the strength of feedback inhibition is close to the strength of feedback excitation at these activities.

Proof: To simplify notation, let $w_i = x_i - \Gamma_i^-$ and define $S_i^*(w) = S_i(x)$. Now multiply $S_i^*(w)$ by the denominator of F_i to find

$$U_i(w) = (q_i^{N_i} + z_i^{N_i}) S_i^*(w). \quad (72)$$

Function $U_i(w) = 0$ at some $w \in \mathbb{R}_+^n$ iff $S_i(x) = 0$ at a suprathreshold value of x , $i = 1, 2, \dots, n$. Use inequality (68) to analytically continue $U_i(w)$ to a function $\tilde{U}_i(z)$ analytic on the polydisk $\Omega = \{z : |z_i| < \epsilon\}$. (In fact, we could define $\tilde{U}_i(z)$ analytically for $|z_i| < r_i$.) Inequality (68) guarantees that all real suprathreshold zeros are included in Ω . We will show the subvariety W of zeros $\tilde{U}_i(z) = 0$, $i = 1, 2, \dots, n$, is contained in the polydisk $\Omega' = \{z : |z_i| < q_i\}$. By (68), $q_i < \epsilon$, $i = 1, 2, \dots, n$. Hence the subvariety W is compact, and the theorem will follow.

To complete the proof, we write $\tilde{U}_i(z)$ in the following form using the notation $R_i(z)$ for the sum of inhibitory feedback terms that analytically continue $\sum_{k=1}^n c_{ik} G_k(w_k +$

Γ_k^-):

$$\begin{aligned} \tilde{U}_i(z) = & -z_i^{N_i-1} [\gamma_i + p_i + R_i(z)] \\ & + z_i^{N_i} [\alpha_i - \gamma_i \Gamma_i^- + p_i (\beta_i - \Gamma_i^-) - \Gamma_i^- R_i(z)] \\ & - z_i^{N_i} [\gamma_i + \Gamma_i^- R_i(z)] \\ & + q_i^{N_i} [\alpha_i - \gamma_i \Gamma_i^- - \Gamma_i^- R_i(z)]. \end{aligned} \quad (73)$$

The analytic continuation $\tilde{G}_k(z_k)$ of $G_k(w_k + \Gamma_k^-)$ can be rewritten as

$$\tilde{G}_k(z_k) = \frac{1}{r_k^{M_k} z_k^{-M_k} + 1}. \quad (74)$$

Because $|z_k| \leq \epsilon$, (68) implies

$$|\tilde{G}_k(z_k)| \leq (\delta^{M_k} - 1)^{-1}. \quad (75)$$

Since (75) is true for every z_k when $z \in \Omega$, it follows for every $i = 1, 2, \dots, n$ that

$$|R_i(z)| \leq s_i \quad \text{if } z \in \Omega. \quad (76)$$

By (73) and (75), if $z \in \Omega$

$$|\tilde{U}_i(z)| \geq L_i(|z_i|) \quad (77)$$

where

$$\begin{aligned} L_i(\xi) = & \xi^{N_i+1} (\gamma_i + p_i - s_i) \\ & - \xi^{N_i} [|\alpha_i - \gamma_i \Gamma_i^-| + p_i |\beta_i - \Gamma_i^-| + \Gamma_i^- s_i] \\ & - \xi^{N_i} [\gamma_i + s_i] - q_i^{N_i} [|\alpha_i - \gamma_i \Gamma_i^-| + \Gamma_i^- s_i]. \end{aligned} \quad (78)$$

To show that $L_i(|z_i|) > 0$ if $\epsilon > |z_i| \geq q_i$, we verify that $L_i(q_i) > 0$ and $(dL_i/d\xi)(\xi) \geq 0$ for $\epsilon > \xi \geq q_i$ using (71). This fact along with (76) completes the proof.

Inequality (68) requires that $q_i < r_i$. Analogous results hold even if $q_i \geq r_i$ when both q_i and r_i are chosen sufficiently large. We state without proof such a theorem.

Theorem 8 (Sigmoid Signals): Suppose that $\epsilon > 0$ and $\delta > 1$ exist such that

$$\max_i (b_i + c_i - \gamma_i, v_i) < \epsilon < \delta\epsilon < \min_{j,k} (q_k, r_k) \quad (79)$$

where

$$v_i = \frac{|\alpha_i - \gamma_i \Gamma_i^-| + \beta_i t_i + \Gamma_i^- (s_i + t_i)}{\gamma_i - (s_i + t_i)}, \quad (80)$$

s_i is defined as in (69),

$$t_i = (\delta^{N_i} - 1)^{-1}, \quad (81)$$

and

$$\gamma_i > s_i + t_i, \quad (82)$$

$i = 1, 2, \dots, n$. Then there are at most finitely many suprathreshold equilibrium points of (57).

Because not all parameter choices of the sigmoid signal functions (66) and (67) have been shown to imply global pattern formation, it is inappropriate to summarize Theorems 7 and 8 as absolute stability results. Instead we summarize the constraints which have been shown to yield

global pattern formation when these sigmoid signal functions are used.

Corollary 5 (Sigmoid Global Pattern Formation): Let system (57) possess a nonnegative symmetric interaction matrix $\|c_{ij}\|$, positive decay rates A_i , and suprathreshold sigmoid signal functions (66) and (67) that satisfy the constraints of Theorem 7 or 8 and the inequalities $M_i > 1$ in (67), $i = 1, 2, \dots, n$. Then global pattern formation occurs.

Proof: The new constraint $M_i > 1$ implies that d_i is differentiable even when $x_i = \Gamma_i^-$, as is required by Theorem 1. The constraint of Theorem 3 that b_i possess no nontrivial level intervals can be violated in (30) only if

$$A_i C_i + (B_i + C_i) I_i = 0. \quad (83)$$

Since $A_i > 0$, this case can only occur if $C_i = 0 = I_i$, which implies that x_i remains between 0 and B_i . Suppose $\Gamma_i^- = 0$. Then all $x_i > 0$ are suprathreshold values, and x_i can attain only one subthreshold equilibrium value, namely zero. Suppose $\Gamma_i^- > 0$. If $x_i(T) \leq \Gamma_i^-$ for some $t = T$, then $x_i(t) \leq \Gamma_i^-$ for all $t = T$. This is true because the excitatory threshold of F_i in (66) equals the inhibitory threshold Γ_i^- of G_i in (67), no input I_i can excite x_i due to (83), and all other v_k , $k \neq i$, can only inhibit x_i . Thus for $t \geq T$, $\dot{x}_i \leq -A_i x_i$, so that x_i approaches the unique subthreshold value zero. In all cases, only one subthreshold equilibrium value of each x_i can exist, which completes the proof.

IX. CONCLUDING REMARKS

The present article notes that systems (1) that are competitive and possess symmetric interactions admit a global Liapunov function. Given this observation, it remains to characterize the set E and its relationship to the equilibrium points of (1). Despite useful partial results, this approach has not yet handled all of the physically interesting neural networks wherein absolute stability may be conjectured to occur. For example, extensive numerical analysis of neural networks of the form

$$\dot{x}_i = -A_i x_i + (B_i - C_i x_i) \left[I_i + \sum_{k=1}^n D_{ik} f_k(x_k) \right] - (E_i x_i + F_i) \left[J_i + \sum_{k=1}^n G_{ik} g_k(x_k) \right] \quad (84)$$

where both matrices $D = \|D_{ik}\|$ and $G = \|G_{ik}\|$ are symmetric suggests that an absolute stability result should exist for these networks, which generalize (3) [3], [5], [31]. In these networks, cooperative interactions $\sum_{k=1}^n D_{ik} f_k(x_k)$ as well as competitive interactions $\sum_{k=1}^n G_{ik} g_k(x_k)$ are permissible. A global Liapunov function whose equilibrium set can be effectively analyzed has not yet been discovered for the networks (84).

It remains an open question whether the Liapunov function approach, which requires a study of equilibrium points, or an alternative global approach, such as the Liapunov functional approach which sidesteps a direct study of equi-

librium points [14], [18], [21], will ultimately handle all of the physically important cases.

REFERENCES

- [1] G. A. Carpenter, "Bursting phenomena in excitable membranes," *SIAM J. Appl. Math.*, vol. 36, pp. 334-372, 1979.
- [2] ———, "Normal and abnormal signal patterns in nerve cells," in *Mathematical Psychology and Psychophysiology*, S. Grossberg, Ed. Providence, RI: Amer. Math. Soc., 1981, pp. 48-90.
- [3] M. A. Cohen and S. Grossberg, "Some global properties of binocular resonances: Disparity matching, filling-in, and figure ground synthesis," in *Figural Synthesis*, T. Caelli and P. Dodwell, Eds. Hillsdale, NJ: Erlbaum Press, 1983.
- [4] M. Eigen and P. Schuster, "The hypercycle: A principle of natural self-organization. B. The abstract hypercycle," *Naturwissenschaften*, vol. 65, pp. 7-41, 1978.
- [5] S. A. Ellis and S. Grossberg, "Pattern formation, contrast control, and oscillations in the short term memory of shunting on-center off-surround networks," *Biol. Cybern.*, vol. 20, pp. 69-98, 1975.
- [6] M. E. Gilpin and F. J. Ayala, "Global models of growth and competition," *Proc. Nat. Acad. Sci.*, vol. 70, pp. 3590-3593, 1973.
- [7] B. S. Goh and T. T. Agnew, "Stability in Gilpin and Ayala's models of competition," *J. Math. Biol.*, vol. 4, pp. 275-279, 1977.
- [8] S. Grossberg, "Some physiological and biochemical consequences of psychological postulates," *Proc. Nat. Acad. Sci.*, vol. 60, pp. 758-765, 1968.
- [9] ———, "On learning information, lateral inhibition, and transmitters," *Math. Biosci.*, vol. 4, pp. 225-310, 1969.
- [10] ———, "Neural pattern discrimination," *J. Theoret. Biol.*, vol. 27, pp. 291-337, 1970.
- [11] ———, "A neural theory of punishment and avoidance. II. Quantitative theory," *Math. Biosci.*, vol. 15, pp. 39-67, 1972.
- [12] ———, "Contour enhancement, short term memory, and constancies in reverberating neural networks," *Studies in Appl. Math.*, vol. 52, pp. 217-257, 1973.
- [13] ———, "On the development of feature detectors in the visual cortex with applications to learning and reaction diffusion systems," *Biol. Cybern.*, vol. 21, pp. 145-159, 1976.
- [14] ———, "Competition, decision, and consensus," *J. Math. Anal. Appl.*, vol. 66, pp. 470-493, 1978.
- [15] ———, "A theory of human memory: Self-organization and performance of sensory-motor codes, maps, and plans," in *Progress Theoretical Biology*, vol. 5, R. Rosen and F. Snell, Eds. New York: Academic, 1978.
- [16] ———, "Communication, memory, and development," in *Progress in Theoretical Biology*, vol. 5, R. Rosen and F. Snell, Eds. New York: Academic, 1978.
- [17] ———, "Decisions, patterns, and oscillations in the dynamics of competitive systems with applications to Volterra-Lotka systems," *J. Theoret. Biol.*, vol. 73, pp. 101-130, 1978.
- [18] ———, "Biological competition: Decision rules, pattern formation, and oscillations," *Proc. Nat. Acad. Sci.*, vol. 77, pp. 2338-2342, 1980.
- [19] ———, "How does a brain build a cognitive code?" *Psychol. Rev.*, vol. 58, pp. 1-51, 1980.
- [20] ———, "Intracellular mechanisms of adaptation and self-regulation in self-organizing networks: The role of chemical transducers," *Bull. Math. Biol.*, vol. 42, 1980.
- [21] ———, "Adaptive resonance in development, perception, and cognition," in *Mathematical Psychology and Psychophysiology*, S. Grossberg, Ed. Providence RI: Amer. Math. Soc., 1981.
- [22] J. Hale, *Ordinary Differential Equations*. New York: Wiley-Interscience, 1969.
- [23] M. W. Hirsch, *Differential Topology*. New York: Springer-Verlag, 1976.
- [24] A. L. Hodgkin, *The Conduction of the Nervous Impulse*. Liverpool, England: Liverpool Univ. Press, 1964.
- [25] J. Kaplan and J. Yorke, "Competitive exclusion and nonequilibrium co-existence," *Amer. Naturalist*, vol. 111, pp. 1031-1036, 1977.
- [26] B. Katz, *Nerve, Muscle, and Synapse*. New York: McGraw-Hill, 1966.
- [27] Kilmer, W. L., "On some realistic constraints in prey-predator

- mathematics." *J. Theoret. Biol.*, vol. 36, pp. 9-22, 1972.
- [28] S. W. Kuffler and J. G. Nicholls. *From Neuron to Brain*. Sunderland, MA: Sinauer Assoc., 1976.
- [29] J. P. LaSalle. "An invariance principle in the theory of stability," in *Differential Equations and Dynamical Systems*, J. K. Hale and J. P. LaSalle, Eds. New York: Academic, 1967.
- [30] ———. "Stability theory for ordinary differential equations." *J. Differential Equations*, vol. 4, pp. 57-65, 1968.
- [31] D. Levine and S. Grossberg, "On visual illusions in neural networks: Line neutralization, tilt aftereffect, and angle expansion." *J. Theoret. Biol.*, vol. 61, pp. 477-504, 1976.
- [32] R. H. MacArthur, "Species packing and competitive equilibrium for many species," *Theoret. Population Biol.*, vol. 1, pp. 1-11, 1970.
- [33] R. M. May and W. J. Leonard. "Nonlinear aspects of competition between three species." *SIAM J. Appl. Math.*, vol. 29, pp. 243-253, 1975.
- [34] F. Ratliff. *Mach Bands: Quantitative Studies of Neural Networks in the Retina*. San Francisco, CA: Holden-Day, 1965.
- [35] W. Rudin. *Function Theory on the Unit Ball of C^n* . New York: Springer-Verlag, 1980.
- [36] P. Schuster, K. Sigmund, and R. Wolff. "On ω -limits for competition between three species." *SIAM J. Appl. Math.*, vol. 37, pp. 49-54, 1979.
- [37] Y. Takeuchi, N. Adachi, and H. Tokumaru. "The stability of generalized Volterra equations." *J. Math. Anal. Appl.*, vol. 62, pp. 453-473, 1978.