Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2011, Article ID 569141, 16 pages
doi:10.1155/2011/569141

Research Article

Hopf Bifurcation Analysis for the van der Pol Equation with Discrete and Distributed Delays

Xiaobing Zhou,[1] Murong Jiang,[1] and Xiaomei Cai[2]

[1] *School of Information Science and Engineering, Yunnan University, Kunming 650091, China*
[2] *Bureau of Asset Management, Yunnan University, Kunming 650091, China*

Correspondence should be addressed to Xiaobing Zhou, zhouxb.cn@gmail.com

Received 5 December 2010; Revised 7 February 2011; Accepted 2 March 2011

Academic Editor: Leonid Shaikhet

We consider the van der Pol equation with discrete and distributed delays. Linear stability of this equation is investigated by analyzing the transcendental characteristic equation of its linearized equation. It is found that this equation undergoes a sequence of Hopf bifurcations by choosing the discrete time delay as a bifurcation parameter. In addition, the properties of Hopf bifurcation were analyzed in detail by applying the center manifold theorem and the normal form theory. Finally, some numerical simulations are performed to illustrate and verify the theoretical analysis.

1. Introduction

Since its introduction in 1927, the van der Pol equation [1] has served as a basic model of self-excited oscillations in physics, electronics, biology, neurology and other disciplines [2–15]. The intensively studied van der Pol equation is governed by the following second-order nonlinear damped oscillatory system:

$$\begin{aligned} \dot{x}(t) &= y(t) - f(x(t)), \\ \dot{y}(t) &= -x(t), \end{aligned} \tag{1.1}$$

where $f(x) = ax + bx^3$, a and b are real constants.

In 1999, Murakami [16] introduced a discrete time delay into system (1.1), and obtained the following pair of delay differential equations

$$\begin{aligned} \dot{x}(t) &= y(t-\tau) - f(x(t-\tau)), \\ \dot{y}(t) &= -x(t-\tau). \end{aligned} \tag{1.2}$$

By using the center manifold theorem, he [16] found that periodic solutions existed in system (1.2). The stability of bifurcating periodic solutions was discussed in detail by Yu and Cao [17].

It is well known that dynamical systems with distributed delay are more general than those with discrete delay. So Liao et al. [18] proposed the following van der Pol equation with distributed delay:

$$\begin{aligned} \dot{x}(t) &= \int_0^{\infty} F(t-s)y(s)\mathrm{d}s - f\left[\int_0^{\infty} F(t-s)x(s)\mathrm{d}s\right], \\ \dot{y}(t) &= -\int_0^{\infty} F(t-s)x(s)\mathrm{d}s. \end{aligned} \tag{1.3}$$

The existence of Hopf bifurcation and the stability of the bifurcating periodic solutions of system (1.3) were analyzed in [18–25] for the weak and strong kernels, respectively.

In [26], Liao considered the following system with two discrete time delays:

$$\begin{aligned} \dot{x}(t) &= y(t-\tau_2) - f(x(t-\tau_1)), \\ \dot{y}(t) &= -x(t-\tau_1). \end{aligned} \tag{1.4}$$

By choosing one of the delays as a bifurcation parameter, system (1.4) was found to undergo a sequence of Hopf bifurcations. The author had also found that resonant codimension two bifurcation occurred in this system.

In this paper, we consider the following van der Pol equation with discrete and distributed time delays:

$$\begin{aligned} \dot{x}(t) &= \int_{-\infty}^{t} F(t-s)y(s)ds - f(x(t-\tau)), \\ \dot{y}(t) &= -x(t-\tau), \end{aligned} \tag{1.5}$$

with initial conditions $x(\theta_1) = \varphi_1(\theta_1)$, $y(\theta_2) = \varphi_2(\theta_2)$, $(-\tau \le \theta_1 \le 0,\ -\infty < \theta_2 \le 0)$, $\tau \ge 0$, where $\varphi_1(\theta_1)$ and $\varphi_2(\theta_2)$ are bounded and are continuous functions. The weight function $F(s)$ is a nonnegative bounded function, which describes the influence of the past states on the current dynamics. It is assumed that the presence of the distributed time delay does not affect the system equilibrium. Hence, $O(0,0)$ is the unique equilibrium of system (1.5).

We normalize the kernel in the following way:

$$\int_0^\infty F(s)\mathrm{d}s = 1. \tag{1.6}$$

Usually, the following form:

$$F(s) = \frac{\alpha^{p+1}s^p e^{-\alpha s}}{p!}, \quad p = 0, 1, 2, \ldots, \tag{1.7}$$

is taken as the kernel. The kernel is called "weak" when $p = 0$ and "strong" when $p = 1$, respectively. The analysis of weak and strong kernels is similar, so we only consider the weak kernel in this paper, that is,

$$F(s) = \alpha e^{-\alpha s}, \quad \alpha > 0, \tag{1.8}$$

where α reflects the mean delay of the weak kernel.

The purpose of this paper is to discuss the stability and bifurcation of system (1.5), which is an extension of the aforementioned systems. By taking the discrete delay τ as the bifurcation parameter, we will show that the equilibrium of system (1.5) loses its stability and Hopf bifurcation occurs when τ passes through a certain critical value.

The remainder of this paper is organized as follows. In Section 2, the linear stability of system (1.5) is discussed and some sufficient conditions for the existence of Hopf bifurcations are derived. The properties of Hopf bifurcation are analyzed in detail by using the center manifold theorem and the normal form theory in Section 3. In Section 4, some numerical simulations are performed to illustrate and verify the theoretical analysis. Finally, conclusions are drawn in Section 5.

2. Linear Stability and Existence of Hopf Bifurcation

In this section, we discuss the linear stability of the equilibrium $O(0,0)$ of system (1.5) and the existence of Hopf bifurcations. For analysis convenience, we define the following variable:

$$z(t) = \int_{-\infty}^{t} \alpha e^{-\alpha(t-s)} y(s)\mathrm{d}s. \tag{2.1}$$

Then by the linear chain trick technique, system (1.5) can be transformed into the following system with only discrete time delay:

$$\begin{aligned} \dot{x}(t) &= z(t) - f(x(t-\tau)), \\ \dot{y}(t) &= -x(t-\tau), \\ \dot{z}(t) &= \alpha y(t) - \alpha z(t). \end{aligned} \tag{2.2}$$

It is obvious that system (2.2) has a unique equilibrium $O(0,0,0)$.

The linearization of system (2.2) at the equilibrium $O(0,0,0)$ is

$$\begin{aligned}\dot{x}(t) &= z(t) - ax(t-\tau),\\ \dot{y}(t) &= -x(t-\tau),\\ \dot{z}(t) &= \alpha y(t) - \alpha z(t).\end{aligned} \tag{2.3}$$

The associated characteristic equation of (2.3) is

$$\det\begin{pmatrix}\lambda + ae^{-\lambda\tau} & 0 & -1\\ e^{-\lambda\tau} & \lambda & 0\\ 0 & -\alpha & \lambda+\alpha\end{pmatrix} = 0, \tag{2.4}$$

which is equivalent to

$$\lambda^3 + \alpha\lambda^2 + \left(a\lambda^2 + a\alpha\lambda + \alpha\right)e^{-\lambda\tau} = 0. \tag{2.5}$$

In the following, we investigate the distribution of roots of (2.5) and obtain the conditions under which system (2.2) undergoes Hopf bifurcation.

We know that $i\omega$ $(\omega > 0)$ is a root of (2.5) if and only if ω satisfies

$$-\omega^3 i - \alpha\omega^2 + \left(-a\omega^2 + a\alpha\omega i + \alpha\right)(\cos\omega\tau - i\sin\omega\tau) = 0. \tag{2.6}$$

Separating the real and imaginary parts, yields

$$\begin{aligned}\alpha\omega^2 &= \left(-a\omega^2 + \alpha\right)\cos\omega\tau + a\alpha\omega\sin\omega\tau,\\ -\omega^3 &= \left(-a\omega^2 + \alpha\right)\sin\omega\tau - a\alpha\omega\cos\omega\tau.\end{aligned} \tag{2.7}$$

Taking square on both sides of the equations in system (2.7) and adding them up yield

$$\omega^6 + \left(\alpha^2 - a^2\right)\omega^4 + \left(2a\alpha - a^2\alpha^2\right)\omega^2 - \alpha^2 = 0. \tag{2.8}$$

Let $z = \omega^2$, $p = \alpha^2 - a^2$, $q = 2a\alpha - a^2\alpha^2$, and $r = -\alpha^2$. Then (2.8) becomes

$$z^3 + pz^2 + qz + r = 0. \tag{2.9}$$

Denote

$$h(z) = z^3 + pz^2 + qz + r. \tag{2.10}$$

Since $\alpha > 0$, then $r = -\alpha^2 < 0$, and $\lim_{z\to\infty} h(z) = \infty$, we can conclude that (2.9) has at least one positive root.

Without loss of generality, we assume that (2.9) has three positive roots, defined by z_1, z_2, and z_3, respectively. Then (2.8) has three positive roots

$$\omega_1 = \sqrt{z_1}, \qquad \omega_2 = \sqrt{z_2}, \qquad \omega_3 = \sqrt{z_3}. \tag{2.11}$$

From (2.7), we have that

$$\cos \omega\tau = \frac{\alpha^2\omega^2}{\left(a\omega^2 - \alpha\right)^2 + a^2\alpha^2\omega^2}. \tag{2.12}$$

Thus, if we denote

$$\tau_k^{(j)} = \frac{1}{\omega_k}\left\{\cos^{-1}\left(\frac{\alpha^2\omega_k^2}{\left(a\omega_k^2 - \alpha\right)^2 + a^2\alpha^2\omega_k^2}\right) + 2j\pi\right\}, \tag{2.13}$$

where $k = 1,2,3$, and $j = 0,1,\ldots$, then $\pm i\omega_k$ is a pair of purely imaginary roots of (2.5) with $\tau_k^{(j)}$. Define

$$\tau_0 = \tau_{k_0}^{(0)} = \min_{k\in\{1,2,3\}}\left\{\tau_k^{(0)}\right\}, \qquad \omega_0 = \omega_{k_0}. \tag{2.14}$$

In order to further investigate (2.5), we need to introduce a result proposed by Ruan and Wei [27], which is stated as follows.

Lemma 2.1 (see [27]). *Consider the exponential polynomial*

$$\begin{aligned} P\left(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m}\right) = \lambda^n &+ p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\ &+ \left[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right]e^{-\lambda\tau_1} + \cdots \\ &+ \left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right]e^{-\lambda\tau_m}, \end{aligned} \tag{2.15}$$

where $\tau_i \geq 0$ $(i = 1,2,\ldots,m)$ and $p_j^{(i)}$ $(i = 0,1,\ldots,m;\ j = 1,2,\ldots,n)$ are constants. As $(\tau_1, \tau_2, \ldots, \tau_m)$ vary, the sum of the order of the zeros of $P(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m})$ on the right half plane can change only if a zero appears on or crosses the imaginary axis.

By applying Lemma 2.1, one can easily obtain the following result on the distribution of roots of (2.5).

Lemma 2.2. *For the third-degree transcendental* (2.5)*, if $r < 0$, then all roots with positive real parts of* (2.5) *have the same sum to those of the polynomial* (2.5) *for $\tau \in [0, \tau_0)$.*

Let

$$\lambda(\tau) = \xi(\tau) + i\omega(\tau) \tag{2.16}$$

be the root of (2.5) near $\tau = \tau_k^{(j)}$, satisfying

$$\xi\left(\tau_k^{(j)}\right) = 0, \qquad \omega\left(\tau_k^{(j)}\right) = \omega_k. \tag{2.17}$$

Then the following transversality condition holds.

Lemma 2.3. *Suppose that $z_k = w_k^2$ and $h'(z_k) \neq 0$, where $h(z)$ is defined by* (2.10)*; then*

$$\frac{d\left(\operatorname{Re}\lambda\left(\tau_k^{(j)}\right)\right)}{d\tau} \neq 0, \tag{2.18}$$

and the sign of $d(\operatorname{Re}\lambda(\tau_k^{(j)}))/d\tau$ is consistent with that of $h'(z_k)$.

Proof. The proof is similar to those in [28–32], so we omit it here.

When $\tau = 0$, (2.5) becomes

$$\lambda^3 + (a+\alpha)\lambda^2 + a\alpha\lambda + \alpha = 0. \tag{2.19}$$

According to the Routh-Hurwitz criterion, if the following conditions:

$$a + \alpha > 0, \qquad a(a+\alpha) > 1, \tag{2.20}$$

hold, then all roots of (2.19) have negative real parts, which means that the equilibrium $O(0,0,0)$ of system (2.19) is stable. □

By applying Lemmas 2.2 and 2.3 to (2.5), we have the following theorem.

Theorem 2.4. *Let $\tau_k^{(j)}$, ω_0, τ_0 and be defined by* (2.13) *and* (2.14)*, respectively. Suppose that $a+\alpha > 0$ and $a(a+\alpha) > 1$. Then one has the following.*

(i) *If $r < 0$, then the equilibrium $O(0,0,0)$ of system* (2.2) *is asymptotically stable for $\tau \in [0, \tau_0)$.*

(ii) *If $r < 0$ and $h'(z_k) \neq 0$, then system* (2.2) *undergoes a Hopf bifurcation at its equilibrium $O(0,0,0)$ when $\tau = \tau_k^{(j)}$.*

3. The Properties of Hopf Bifurcation

In the previous section, we obtain some conditions for Hopf bifurcations to occur at the critical value $\tau_k^{(j)}$. In this section, we analyze the properties of the Hopf bifurcation by virtue of the method proposed by Hassard et al. [33], namely, to determine the direction of Hopf bifurcation and the stability of bifurcating periodic solutions bifurcating from the equilibrium $O(0,0,0)$ for system (2.2) by applying the normal form theory and the center manifold theorem.

For analysis convenience, let $t = s\tau$, $x_1(s) = x(s\tau)$, $y_1(s) = y(s\tau)$, $z_1(s) = z(s\tau)$, and $\tau = \tau_k^{(j)} + \mu$, $\mu \in R$. Denote $t = s$; then system (2.2) is transformed into the following form:

$$\begin{aligned}
\dot{x}_1(t) &= \left(\tau_k^{(j)} + \mu\right)(z_1(t) - f(x_1(t-1))), \\
\dot{y}_1(t) &= \left(\tau_k^{(j)} + \mu\right)(-x_1(t-1)), \\
\dot{z}_1(t) &= \left(\tau_k^{(j)} + \mu\right)(\alpha y_1(t) - \alpha z_1(t)),
\end{aligned} \tag{3.1}$$

where $f(x_1(t-1)) = ax_1(t-1) + bx_1^3(t-1)$.

Its linear part is

$$\begin{aligned}
\dot{x}_1(t) &= \left(\tau_k^{(j)} + \mu\right)(z_1(t) - ax_1(t-1)), \\
\dot{y}_1(t) &= \left(\tau_k^{(j)} + \mu\right)(-x_1(t-1)), \\
\dot{z}_1(t) &= \left(\tau_k^{(j)} + \mu\right)(\alpha y_1(t) - \alpha z_1(t)),
\end{aligned} \tag{3.2}$$

and the nonlinear part is as follows:

$$f(\mu, x_t) = \left(\tau_k^{(j)} + \mu\right)\begin{pmatrix} -bx_1^3(t-1) \\ 0 \\ 0 \end{pmatrix}. \tag{3.3}$$

Denote $C^k[-1,0] = \{\phi \mid \phi : [-1,0] \to R^3$, each component of ϕ has k-order continuous derivative$\}$. For $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta))^T \in C^0[-1,0]$, we define

$$L_\mu \phi = \left(\tau_k^{(j)} + \mu\right)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & \alpha & -\alpha \end{pmatrix}\phi(0) + \left(\tau_k^{(j)} + \mu\right)\begin{pmatrix} -a & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\phi(-1), \tag{3.4}$$

where L_μ is a one-parameter family of bounded linear operators in $C^0[-1,0] \to R^3$. By the Riesz representation theorem, there exists a 3×3 matrix whose components are bounded variation functions $\eta(\theta, \mu)$ in $[-1,0] \to R^3$ such that

$$L_\mu \phi = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta). \tag{3.5}$$

In fact, we can choose

$$\eta(\theta,\mu)=\left(\tau_k^{(j)}+\mu\right)\begin{pmatrix}0&0&1\\0&0&0\\0&\alpha&-\alpha\end{pmatrix}\delta(\theta)-\left(\tau_k^{(j)}+\mu\right)\begin{pmatrix}-a&0&0\\-1&0&0\\0&0&0\end{pmatrix}\delta(\theta+1),\tag{3.6}$$

where δ is the Dirac delta function, which is defined by $\delta(\theta)=\begin{cases}0, & \theta\neq 0,\\ 1, & \theta=0.\end{cases}$

For $\phi\in C^1([-1,0],R^3)$, define

$$A(\mu)\phi=\begin{cases}\dfrac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}, & \theta\in[-1,0),\\ \displaystyle\int_{-1}^{0}\mathrm{d}\eta(\mu,s)\phi(s), & \theta=0,\end{cases}$$
$$R(\mu)\phi=\begin{cases}0, & \theta\in[-1,0),\\ f(\mu,\phi), & \theta=0.\end{cases}\tag{3.7}$$

Then, we can transform system (2.2) into an operator equation as the following form:

$$\dot{x}_t=A(\mu)x_t+R(\mu)x_t,\tag{3.8}$$

where $x_t(\theta)=(x_1(t+\theta),y_1(t+\theta),z_1(t+\theta))^T$ for $\theta\in[-1,0]$.

For $\psi\in C^1([0,1],(R^3)^*)$, define

$$A^*\psi(s)=\begin{cases}-\dfrac{\mathrm{d}\psi(s)}{\mathrm{d}s}, & s\in(0,1],\\ \displaystyle\int_{-1}^{0}\psi(-t)\mathrm{d}\eta(t,0), & s=0\end{cases}\tag{3.9}$$

and a bilinear form

$$\langle\psi(s),\phi(\theta)\rangle=\overline{\psi}(0)\phi(0)-\int_{-1}^{0}\int_{\xi=0}^{\theta}\overline{\psi}(\xi-\theta)\mathrm{d}\eta(\theta)\phi(\xi)\mathrm{d}\xi,\tag{3.10}$$

where $\eta(\theta)=\eta(\theta,0)$. Then $A(0)$ and A^* are adjoint operators. By the discussion in Section 2, we know that $\pm i\omega_k\tau_k^{(j)}$ are eigenvalues of $A(0)$, and they are also eigenvalues of A^*.

Suppose that $q(\theta) = (1, \beta, \gamma)^T e^{i\theta\omega_k \tau_k^{(j)}}$ is the eigenvector of $A(0)$ corresponding to $i\tau_k^{(j)}\omega_k$; then $A(0)q(\theta) = i\tau_k^{(j)}\omega_k q(\theta)$. It follows from the definition of $A(0)$ and (3.5) and (3.6) that

$$\tau_k^{(j)} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & \alpha & -\alpha \end{pmatrix} \begin{pmatrix} 1 \\ \beta \\ \gamma \end{pmatrix} + \tau_k^{(j)} \begin{pmatrix} -a & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\omega_k \tau_k^{(j)}} \\ \beta e^{-i\omega_k \tau_k^{(j)}} \\ \gamma e^{-i\omega_k \tau_k^{(j)}} \end{pmatrix} = i\omega_k \tau_k^{(j)} \begin{pmatrix} 1 \\ \beta \\ \gamma \end{pmatrix}. \tag{3.11}$$

Thus, we can easily obtain

$$\beta = \frac{(\alpha + i\omega_k)\left(i\omega_k + ae^{-i\omega_k \tau_k^{(j)}}\right)}{\alpha}, \quad \gamma = i\omega_k + ae^{-i\omega_k \tau_k^{(j)}}. \tag{3.12}$$

On the other hand, suppose that $q^*(s) = B(1, \beta^*, \gamma^*)^T e^{is\omega_k \tau_k^{(j)}}$ is the eigenvector of A^* corresponding to $-i\omega_k \tau_k^{(j)}$. By the definition of A^* and (3.5) and (3.6), we have

$$\tau_k^{(j)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 1 & 0 & -\alpha \end{pmatrix} \begin{pmatrix} 1 \\ \beta^* \\ \gamma^* \end{pmatrix} + \tau_k^{(j)} \begin{pmatrix} -a & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{i\omega_k \tau_k^{(j)}} \\ \beta^* e^{i\omega_k \tau_k^{(j)}} \\ \gamma^* e^{i\omega_k \tau_k^{(j)}} \end{pmatrix} = -i\omega_k \tau_k^{(j)} \begin{pmatrix} 1 \\ \beta^* \\ \gamma^* \end{pmatrix}. \tag{3.13}$$

Therefore, we have

$$\beta^* = \frac{\alpha}{i\omega_k(i\omega_k - \alpha)}, \qquad \gamma^* = \frac{1}{\alpha - i\omega_k}. \tag{3.14}$$

In order to assure $\langle q^*(s), q(\theta) \rangle = 1$, we need to determine the value of B. From (3.10), we have that

$$\begin{aligned}
\langle q^*(s), q(\theta) \rangle &= \overline{q^*}(0)q(0) - \int_{-1}^{0} \int_{\xi=0}^{\theta} \overline{q^*}(\xi - \theta)\mathrm{d}\eta(\theta)q(\xi)\mathrm{d}\xi \\
&= \overline{B}\left(1, \overline{\beta^*}, \overline{\gamma^*}\right)(1, \beta, \gamma)^T - \int_{-1}^{0} \int_{\xi=0}^{\theta} \overline{B}\left(1, \overline{\beta^*}, \overline{\gamma^*}\right) e^{-i(\xi - \theta)\omega_k \tau_k^{(j)}} \mathrm{d}\eta(\theta)(1, \beta, \gamma)^T e^{i\xi\omega_k \tau_k^{(j)}} \mathrm{d}\xi \\
&= \overline{B}\left\{1 + \beta\overline{\beta^*} + \gamma\overline{\gamma^*} - \int_{-1}^{0} \left(1, \overline{\beta^*}, \overline{\gamma^*}\right)\theta e^{i\theta\omega_k \tau_k^{(j)}} \mathrm{d}\eta(\theta)(1, \beta, \gamma)^T\right\} \\
&= \overline{B}\left(1 + \beta\overline{\beta^*} + \gamma\overline{\gamma^*} - \tau_k^{(j)}(a + \beta)e^{-i\omega_k \tau_k^{(j)}}\right).
\end{aligned} \tag{3.15}$$

Thus, we can choose B as

$$B = \frac{1}{1 + \overline{\beta}\beta^* + \overline{\gamma}\gamma^* - \tau_k^{(j)}\left(a + \overline{\beta}\right)e^{i\omega_k \tau_k^{(j)}}}. \tag{3.16}$$

Similarly, we can get $\langle q^*(s), \overline{q}(\theta)\rangle = 0$.

Using the same notation as in Hassard et al. [33], we compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let x_t be the solution of (3.8) when $\mu = 0$.

Define

$$z(t) = \langle q^*, x_t \rangle, \qquad W(t,\theta) = x_t(\theta) - 2\,\mathrm{Re}\{z(t)q(\theta)\}. \tag{3.17}$$

On the center manifold C_0 we have that

$$W(t,\theta) = W(z, \overline{z}, \theta), \tag{3.18}$$

where

$$W(z,\overline{z},\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots, \tag{3.19}$$

in which z and $\overline{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\overline{q}^*$, respectively. Note that W is real if x_t is real. We consider only real solutions.

For the solution $x_t \in C_0$ of (3.8), since $\mu = 0$, we have that

$$\begin{aligned}
\dot{z}(t) &= i\tau_k^{(j)}\omega_k z + \langle \overline{q}^*(\theta), f(0, W(z,\overline{z},\theta) + 2\,\mathrm{Re}\{zq(\theta)\})\rangle \\
&= i\tau_k^{(j)}\omega_k z + \overline{q}^*(0) f(0, W(z,\overline{z},0) + 2\,\mathrm{Re}\{zq(0)\}) \\
&= i\tau_k^{(j)}\omega_k z + \overline{q}^*(0) f_0(z,\overline{z}).
\end{aligned} \tag{3.20}$$

We rewrite this equation as

$$\dot{z}(t) = i\tau_k^{(j)}\omega_k z(t) + g(z,\overline{z}), \tag{3.21}$$

where

$$g(z,\overline{z}) = \overline{q}^*(0) f_0(z,\overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots. \tag{3.22}$$

Noting that $x_t(\theta) = W(t,\theta) + zq(\theta) + \overline{z}\,\overline{q}(\theta)$ and $q(\theta) = (1,\beta,\gamma)^T e^{i\theta\omega_k\tau_k}$, we have that

$$x_1(-1) = z + \overline{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\overline{z} + W_{02}^{(1)}(0)\frac{\overline{z}^2}{2} + O\left(|(z,\overline{z})|^3\right). \tag{3.23}$$

According to (3.20) and (3.21), we know that

$$g(z,\overline{z}) = \overline{q}^*(0)f(z,\overline{z}) = \tau_k^{(j)}\overline{B}\left(1,\overline{\beta^*},\overline{\gamma^*}\right)\begin{pmatrix} -bx_1^3(t-1) \\ 0 \\ 0 \end{pmatrix}, \tag{3.24}$$

where

$$x_1(t+\theta) = W^{(1)}(t,\theta) + z(t)q^{(1)}(\theta) + \overline{z}(t)\overline{q}^{(1)}(\theta). \tag{3.25}$$

From (3.20) and (3.24), we have that

$$\begin{aligned} g(z,\overline{z}) &= -\tau_k^{(j)} b\overline{B}x_1^3(t-1) \\ &= -\tau_k^{(j)} b\overline{B}\left[W^{(1)}(t,\theta) + z(t)q^{(1)}(\theta) + \overline{z}(t)\overline{q}^{(1)}(\theta)\right]^3 \\ &= -\tau_k^{(j)} b\overline{B}\left[W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\overline{z} + W_{02}^{(1)}(-1)\frac{\overline{z}^2}{2} + z(t)q^{(1)}(-1) + \overline{z}(t)\overline{q}^{(1)}(-1)\right]^3. \end{aligned} \tag{3.26}$$

Comparing the coefficients in (3.26) with those in (3.22), we have that

$$\begin{aligned} g_{20} &= 0, \\ g_{11} &= 0, \\ g_{02} &= 0, \\ g_{21} &= -6\tau_k^{(j)} b\overline{B}\left(q^{(1)}(-1)\right)^2\overline{q}^{(1)}(-1). \end{aligned} \tag{3.27}$$

Thus, we can calculate the following values:

$$\begin{aligned} c_1(0) &= \frac{i}{2\tau_k^{(j)}\omega_k}\left(g_{11}g_{20} - 2\left|g_{11}\right|^2 - \frac{\left|g_{02}\right|^2}{3}\right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\left\{\lambda'\left(\tau_k^{(j)}\right)\right\}}, \\ \beta_2 &= 2\operatorname{Re}\{c_1(0)\}, \\ t_2 &= -\frac{\operatorname{Im}\{c_1(0)\} + \mu_2 \operatorname{Im}\left\{\lambda'\left(\tau_k^{(j)}\right)\right\}}{\omega_k\tau_k^{(j)}}, \end{aligned} \tag{3.28}$$

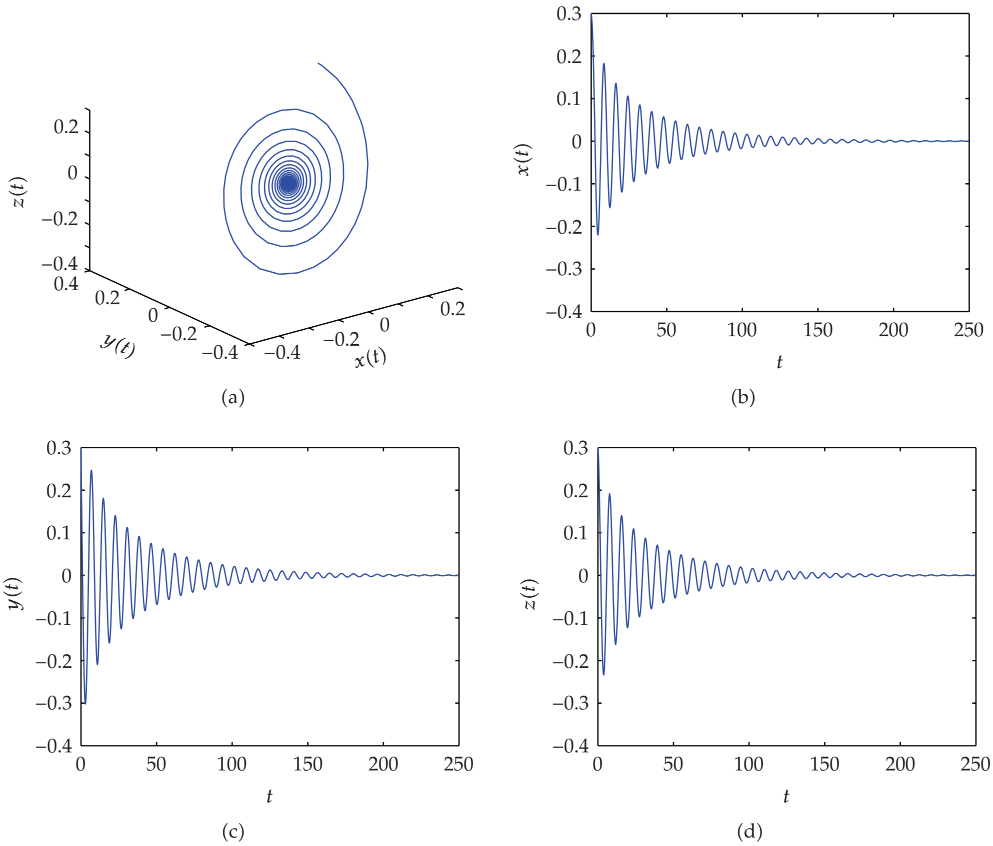

Figure 1: Phase portrait and waveform portraits of system (2.2) with $\tau = 0.37$.

which we need to investigate the properties of Hopf bifurcation. According to [33], we know that μ_2 determines the direction of the Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau > \tau_k^{(j)}$ ($\tau < \tau_k^{(j)}$); β_2 determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions are stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$); t_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $t_2 > 0$ ($t_2 < 0$).

4. A Numerical Example

In this section, we use the formulae obtained in Sections 2 and 3 to verify the existence of a Hopf bifurcation and calculate the Hopf bifurcation value and the direction of the Hopf bifurcation of system (2.2) with $\alpha = 1$, $a = 0.9$, and $b = 2$.

By the results in Section 2, we can determine that

$$z_1 = 0.6506, \qquad \omega_0 = 0.8066, \qquad \tau_0 = 0.4628. \tag{4.1}$$

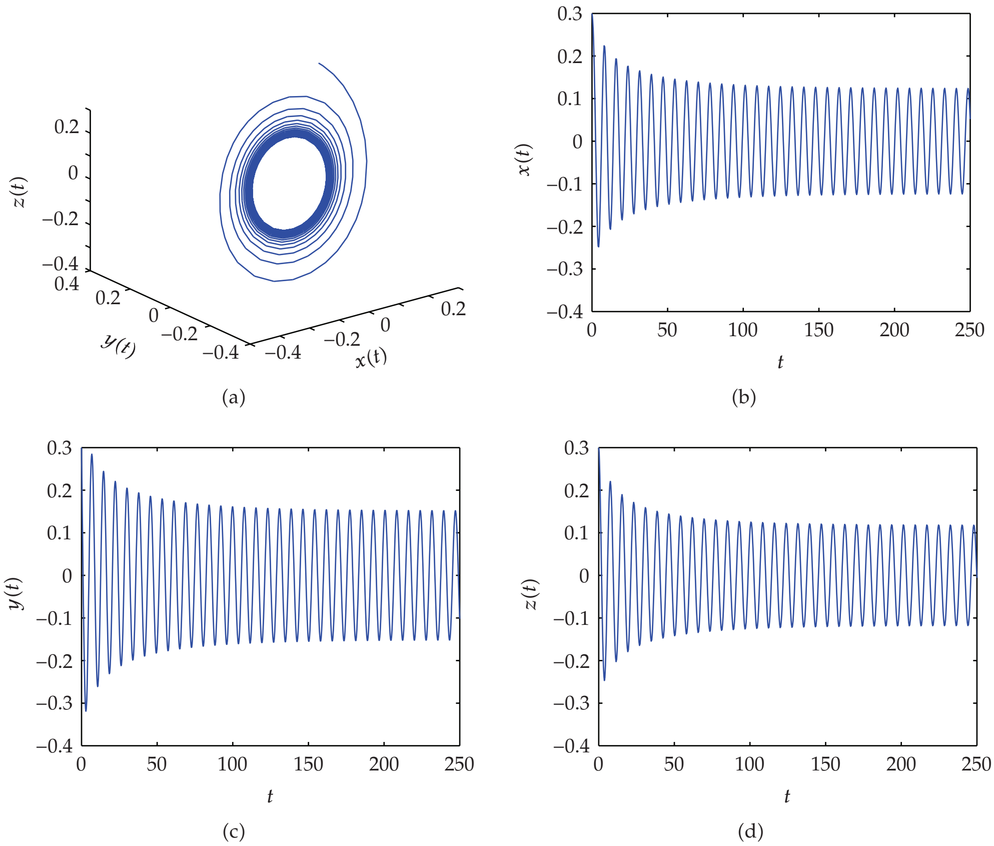

Figure 2: Phase portrait and waveform portraits of system (2.2) with $\tau = 0.5$.

It follows from (3.28) that

$$c_1(0) = -1.2260 - 0.4739i, \qquad \mu_2 = 5.8875,$$
$$\beta_2 = -2.452, \qquad t_2 = -1.9408. \tag{4.2}$$

In light of Theorem 2.4, the equilibrium $O(0,0,0)$ of system (2.2) is stable when $\tau < \tau_0$. This is illustrated in Figure 1 with $\tau = 0.37$. Since $\mu_2 > 0$, when τ passes through the critical value $\tau_0 = 0.4628$, the equilibrium $O(0,0,0)$ loses its stability and a Hopf bifurcation occurs, that is, periodic solutions bifurcate from the equilibrium $O(0,0,0)$. The individual periodic orbits are stable since $\beta_2 < 0$. Figure 2 shows that there are stable limit cycles for system (2.2) with $\tau = 0.5$. Since $t_2 < 0$, the period of the periodic solutions decreases as τ increases. For $\tau = 0.55$, the phase portrait and the waveform portraits are shown in Figure 3. We can tell from Figures 2 and 3 that the period of $\tau = 0.55$ is slightly smaller than that of $\tau = 0.5$.

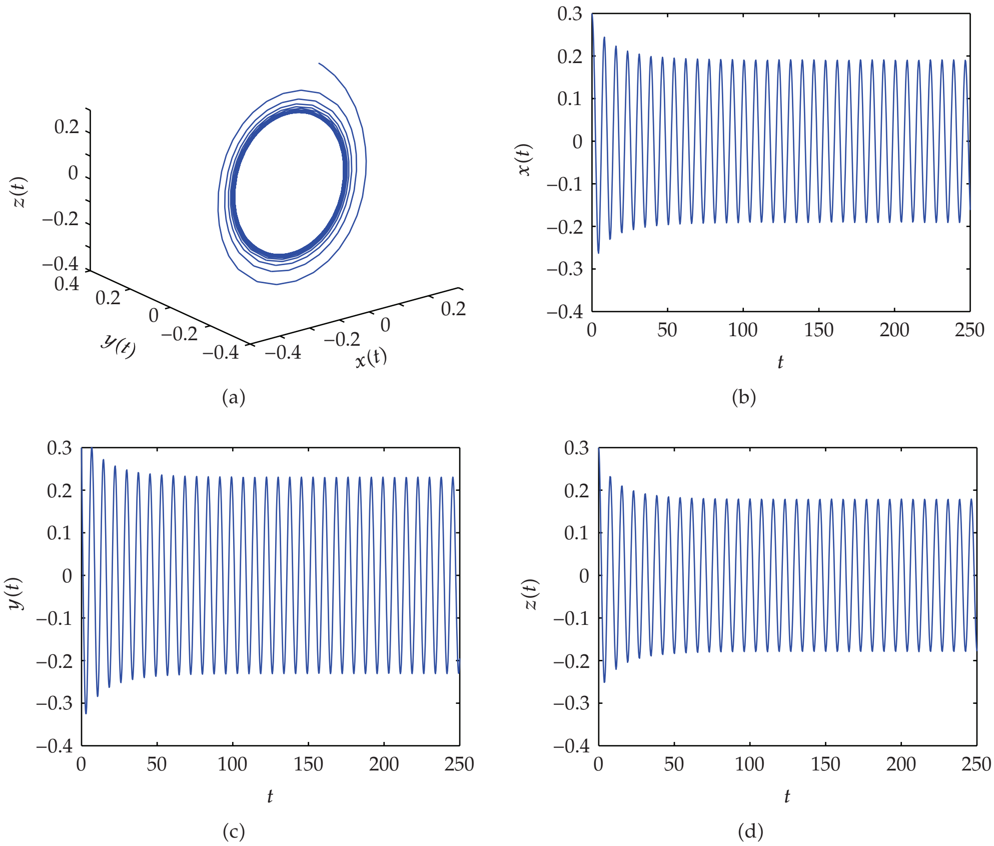

Figure 3: Phase portrait and waveform portraits of system (2.2) with $\tau = 0.55$.

5. Conclusions

The van der Pol equation with discrete and distributed delays is investigated in this paper. Sufficient conditions on the linear stability of this van der Pol equation have been obtained by analyzing the associated transcendental characteristic equation. By choosing the discrete time delay as a bifurcation parameter, we have shown that this equation undergoes a sequence of Hopf bifurcations. In addition, formulae for determining the direction of Hopf bifurcation and the stability of bifurcating periodic solutions are derived. Simulation results have verified and demonstrated the correctness of the theoretical analysis.

Acknowledgments

The authors sincerely thank the reviewers for their helpful comments and suggestions. This work is supported by the Scientific Research Foundation of Yunnan University under Grant 2008YB011, the Natural Science Foundation of Yunnan Province under Grants 2008PY034 and 2009CD019, and the Natural Science Foundation of China under Grants 61065008 and 11026225.

References

[1] B. van der Pol, "Forced oscillations in a circuit with nonlinear resistance (receptance with reactivetriode)," *Philosphical Magazine Series*, vol. 3, pp. 65–80, 1927.

[2] B. Z. Kaplan and I. Yaffe, "An 'improved' van der Pol equation and some of its possible applications," *International Journal of Electronics*, vol. 41, no. 2, pp. 189–198, 1976.

[3] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, vol. 42 of *Applied Mathematical Sciences*, Springer, New York, Ny, USA, 1983.

[4] W. Wang, "Bifurcations and chaos of the Bonhoeffer-van der Pol model," *Journal of Physics A*, vol. 22, no. 13, pp. L627–L632, 1989.

[5] T. Nomura, S. Sato, S. Doi, J. P. Segundo, and M. D. Stiber, "A Bonhoeffer-van der Pol oscillator model of locked and non-locked behaviors of living pacemaker neurons," *Biological Cybernetics*, vol. 69, no. 5-6, pp. 429–437, 1993.

[6] J. C. Chedjou, H. B. Fotsin, and P. Woafo, "Behavior of the Van der Pol oscillator with two external periodic forces," *Physica Scripta*, vol. 55, no. 4, pp. 390–393, 1997.

[7] S. Barland, O. Piro, M. Giudici, J. R. Tredicce, and S. Balle, "Experimental evidence of van der Pol-Fitzhugh-Nagumo dynamics in semiconductor optical amplifiers," *Physical Review E*, vol. 68, no. 3, Article ID 036209, 6 pages, 2003.

[8] M. S. Dutra, A. C. De Pina Filho, and V. F. Romano, "Modeling of a bipedal locomotor using coupled nonlinear oscillators of Van der Pol," *Biological Cybernetics*, vol. 88, no. 4, pp. 286–292, 2003.

[9] I. B. Semenov, Y. V. Mitrishkin, A. A. Subbotin et al., "A van der pol coupled-oscillator model as a basis for developing a system for suppressing MHD instabilities in a tokamak," *Plasma Physics Reports*, vol. 32, no. 2, pp. 114–118, 2006.

[10] L. A. Low, P. G. Reinhall, D. W. Storti, and E. B. Goldman, "Coupled van der Pol oscillators as a simplified model for generation of neural patterns for jellyfish locomotion," *Structural Control and Health Monitoring*, vol. 13, no. 1, pp. 417–429, 2006.

[11] B. Z. Kaplan, I. Gabay, G. Sarafian, and D. Sarafian, "Biological applications of the "Filtered" Van der Pol oscillator," *Journal of the Franklin Institute*, vol. 345, no. 3, pp. 226–232, 2008.

[12] X. K. Ma, M. Yang, J. L. Zou, and L. T. Wang, "Study of complex behavior in a time-delayed van der Pol's electromagnetic system (I)—the phenomena of bifurcations and chaos," *Acta Physica Sinica*, vol. 55, no. 11, pp. 5648–5656, 2006.

[13] X. Li, J. C. Ji, and C. H. Hansen, "Dynamics of two delay coupled van der Pol oscillators," *Mechanics Research Communications*, vol. 33, no. 5, pp. 614–627, 2006.

[14] A. Maccari, "Vibration amplitude control for a van der Pol-Duffing oscillator with time delay," *Journal of Sound and Vibration*, vol. 317, no. 1-2, pp. 20–29, 2008.

[15] M. Belhaq and S. Mohamed Sah, "Fast parametrically excited van der Pol oscillator with time delay state feedback," *International Journal of Non-Linear Mechanics*, vol. 43, no. 2, pp. 124–130, 2008.

[16] K. Murakami, "Bifurcated periodic solutions for delayed van der Pol equation," *Neural, Parallel & Scientific Computations*, vol. 7, no. 1, pp. 1–16, 1999.

[17] W. Yu and J. Cao, "Hopf bifurcation and stability of periodic solutions for van der Pol equation with time delay," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 62, no. 1, pp. 141–165, 2005.

[18] X. Liao, K.-W. Wong, and Z. Wu, "Hopf bifurcation and stability of periodic solutions for van der Pol equation with distributed delay," *Nonlinear Dynamics*, vol. 26, no. 1, pp. 23–44, 2001.

[19] X. Liao, K. Wong, and Z. Wu, "Stability of bifurcating periodic solutions for van der Pol equation with continuous distributed delay," *Applied Mathematics and Computation*, vol. 146, no. 2-3, pp. 313–334, 2003.

[20] S. Li, X. Liao, and S. Li, "Frequency domain approach to hopf bifurcation for Van Der Pol equation with distributed delay," *Latin American Applied Research*, vol. 34, no. 4, pp. 267–274, 2004.

[21] X. Zhou, Y. Wu, Y. Li, and X. Yao, "Stability and Hopf bifurcation analysis on a two-neuron network with discrete and distributed delays," *Chaos, Solitons and Fractals*, vol. 40, no. 3, pp. 1493–1505, 2009.

[22] H. Shu, Z. Wang, and Z. Lü, "Global asymptotic stability of uncertain stochastic bi-directional associative memory networks with discrete and distributed delays," *Mathematics and Computers in Simulation*, vol. 80, no. 3, pp. 490–505, 2009.

[23] W. Su and Y. Chen, "Global robust stability criteria of stochastic Cohen-Grossberg neural networks with discrete and distributed time-varying delays," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 2, pp. 520–528, 2009.

[24] T. Li, Q. Luo, C. Sun, and B. Zhang, "Exponential stability of recurrent neural networks with time-varying discrete and distributed delays," *Nonlinear Analysis: Real World Applications*, vol. 10, no. 4, pp. 2581–2589, 2009.

[25] H. Li, H. Gao, and P. Shi, "New passivity analysis for neural networks with discrete and distributed delays," *IEEE Transactions on Neural Networks*, vol. 21, no. 11, pp. 1842–1847, 2010.

[26] X. Liao, "Hopf and resonant codimension two bifurcation in van der Pol equation with two time delays," *Chaos, Solitons and Fractals*, vol. 23, no. 3, pp. 857–871, 2005.

[27] S. Ruan and J. Wei, "On the zeros of transcendental functions with applications to stability of delay differential equations with two delays," *Dynamics of Continuous, Discrete & Impulsive Systems A*, vol. 10, no. 6, pp. 863–874, 2003.

[28] S. Ruan and J. Wei, "On the zeros of a third degree exponential polynomial with applications to a delayed model for the control of testosterone secretion," *IMA Journal of Mathemathics Applied in Medicine and Biology*, vol. 18, no. 1, pp. 41–52, 2001.

[29] X. Li and J. Wei, "On the zeros of a fourth degree exponential polynomial with applications to a neural network model with delays," *Chaos, Solitons and Fractals*, vol. 26, no. 2, pp. 519–526, 2005.

[30] Y. Song, M. Han, and J. Wei, "Stability and Hopf bifurcation analysis on a simplified BAM neural network with delays," *Physica D*, vol. 200, no. 3-4, pp. 185–204, 2005.

[31] H. Hu and L. Huang, "Stability and Hopf bifurcation analysis on a ring of four neurons with delays," *Applied Mathematics and Computation*, vol. 213, no. 2, pp. 587–599, 2009.

[32] D. Fan, L. Hong, and J. Wei, "Hopf bifurcation analysis in synaptically coupled HR neurons with two time delays," *Nonlinear Dynamics*, vol. 62, no. 1-2, pp. 305–319, 2010.

[33] B. D. Hassard, N. D. Kazarinoff, and Y. H. Wan, *Theory and Applications of Hopf Bifurcation*, vol. 41 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge, UK, 1981.