Applied Mathematics Letters 22 (2009) 1412–1417

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

An application of Razumikhin theorem to exponential stability for linear non-autonomous systems with time-varying delay

Vu N. Phat [a,*], Le V. Hien [b]

[a] Institute of Mathematics, 18 Hoang Quoc Viet Road, Hanoi 10307, Viet Nam
[b] Department of Mathematics, Hanoi National University of Education I, Hanoi, Viet Nam

ARTICLE INFO

Article history:
Received 1 September 2008
Received in revised form 12 December 2008
Accepted 7 January 2009

Keywords:
Linear time-varying system
Exponential stability
Time-varying delay
Razumikhin theorem
Riccati equation

ABSTRACT

In this work, in the light of the Razumikhin stability theorem combined with the Newton–Leibniz formula, a new delay-dependent exponential stability condition is first derived for linear non-autonomous time delay systems without using model transformation and bounding techniques on the derivative of the time-varying delay function. The condition is presented in terms of the solution of Riccati differential equations.

1. Introduction

Recently, special interest has been devoted to the exponential stability problem for linear time delay systems [1–4]. For example, stability conditions for the systems where the time-varying delay function $h(t)$ satisfies

$$h(t) \le h, \quad \dot{h}(t) \le \delta < 1,$$

are given in [5,6]. In [7], the stability conditions have been improved by removing the condition $\delta < 1$. A Razumikhin approach is used in [8,7] for studying the stability of linear autonomous systems without a differentiability assumption on the time delay function and the conditions are presented in terms of linear matrix inequalities (LMIs). It is obvious that the results of these works cannot be extended to non-autonomous time delay systems due to the unsolved infinite systems of LMIs. Some results in [5,6] are extended to linear non-autonomous systems with time-varying delays. Although the results in [5,6] are shown to be less conservative than some existing ones, they still require some restriction on the derivative of the time-varying delay function. To the best of the authors' knowledge, the issue of exponential stability for linear non-autonomous systems without restriction on the derivative of the time delay function remains open, which motivated this work.

In this work, we will consider the problem of exponential stability for linear non-autonomous systems with time-varying delay. The restriction on the derivative of the time delay function is removed, which means that a fast time-varying delay is allowed. On the basis of the Razumikhin theorem combined with the Newton–Leibniz formula, a new delay-dependent exponential stability condition for the system is first derived in terms of the solution of Riccati differential equations, which allow us to compute simultaneously the two bounds that characterize the stability rate of the solution.

* Corresponding author.
E-mail address: vnphat@math.ac.vn (V.N. Phat).

0893-9659/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2009.01.053

2. Preliminaries

We start by introducing some notation and definitions that will be employed throughout the work. R^+ denotes the set of all real non-negative numbers; R^n denotes the n-dimensional space; $\langle x, y\rangle$ or x^Ty denotes the scalar product of two vectors x, y; $\|x\|$ denotes the Euclidean norm of x; $R^{n\times r}$ denotes the space of all $(n \times r)$ matrices; $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\max}(A) = \max\{Re\lambda : \lambda \in \lambda(A)\}$; $\mu(A)$ denotes the matrix measure of A defined by $\mu(A) = \frac{1}{2}\lambda_{\max}(A + A^T)$; $C([-h, 0], R^n)$ denotes the Banach space of all R^n valued continuous functions mapping $[-h, 0]$ into R^n; $BM^+(0, \infty)$ denotes the set of all symmetric positive semi-definite matrix functions bounded in $t \geq 0$.

Consider a linear non-autonomous system with time-varying delay of the form

$$\begin{aligned} &\dot{x}(t) = A(t)x(t) + A_1(t)x(t - h(t)), \quad t \geq 0, \\ &x(t) = \phi(t), \quad t \in [-h, 0], \end{aligned} \tag{2.1}$$

where $h \geq 0$, $x(t) \in R^n$, $A(t), A_1(t) \in R^n$ are given matrix functions, which are continuous and bounded in $t \geq 0$; $\phi(t) \in C([-h, 0], R^n)$ is the initial function with the norm $\|\phi\| = \sup_{s\in[-h,0]} \|\phi(s)\|$. The time delay function $h(t)$ is continuous and satisfies

$$0 \leq h(t) \leq h, \quad \forall t \geq 0.$$

The condition on $h(t)$ means that the upper bound for the time-varying delay is available, and no restriction on the derivative of $h(t)$ is needed, which allows the time delay to be a fast time-varying function.

Definition 2.1. The zero solution of system (2.1) is said to be exponentially stable if there exist positive numbers N, α such that every solution $x(t, \phi)$ of the system satisfies

$$\|x(t, \phi)\| \leq N\|\phi\|e^{-\alpha(t-t_0)}, \quad \forall t \geq t_0 \geq 0.$$

The following results are needed for the proof of the main result.

Proposition 2.1 (*Cauchy Inequality*). *For any symmetric positive definite matrix $W \in R^{n\times n}$ and $x, y \in R^n$, we have*

$$\pm 2\langle x, y\rangle \leq \langle Wx, x\rangle + \langle W^{-1}y, y\rangle.$$

Proposition 2.2 (*Razumikhin Stability Theorem [9]*). *Assume that $u, v, w : R^+ \to R^+$ are nondecreasing, and $u(s), v(s)$ are positive for $s \geq 0$, $v(0) = u(0) = 0$, and $q > 1$. If there is a function $V(t, x) : R^+ \times R^n \to R^+$ such that:*

(i) $u(\|x\|) \leq V(t, x) \leq v(\|x\|)$, $t \in R^+, x \in R^n$,
(ii) $\dot{V}(t, x(t)) \leq -w(\|x(t)\|)$ *if* $V(t + s, x(t + s)) \leq qV(t, x(t))$, $\forall s \in [-h, 0]$, $t \geq 0$,

then the zero solution of system (2.1) is asymptotically stable.

Proposition 2.3 (*[10]*). *For any matrices A, P, E, F, H with appropriate dimensions and $P > 0$, $F^TF \leq I$ and scalar $\rho > 0$, we have:*

(i) $EFH + H^TF^TE^T \leq \rho^{-1}EE^T + \rho H^TH$;
(ii) *if* $\rho I - HPH^T > 0$ *then*

$$(A + EFH)P(A + EFH)^T \leq APA^T + APH^T(\rho I - HPH^T)^{-1}HPA^T + \rho^{-1}EE^T.$$

3. Main result

In this section, sometimes for the sake of brevity, we will omit the arguments of matrix functions if it does not cause any confusion. Given positive numbers $\lambda, h, \beta, \epsilon$ we set

$$P_\beta(t) = P(t) + \beta I, \qquad p = \sup_{t\in R^+} \|P(t)\|,$$

$$a = \sup_{t\in R^+} \|A(t)A^T(t)\|, \qquad a_1 = \sup_{t\in R^+} \|A_1(t)A_1^T(t)\|,$$

$$\mu(A) = \sup_{t\in R^+} \mu(A(t)), \qquad \overline{A}(t) = A(t) + A_1(t),$$

$$\mathcal{A}(t) = \overline{A}(t) + 2h\beta A_1(t)A_1^T(t) + 2h\lambda^{-1}I,$$

$$\gamma = 2\beta\mu(\overline{A}) + 2h\beta^2 a_1 + 2h\lambda^{-1} + \epsilon.$$

Theorem 3.1. *The zero solution of system (2.1) is exponentially stable if there exist positive numbers β, λ, ϵ where $\lambda^{-1}\beta \geq \max\{a, a_1\}$, and a matrix function $P \in BM^+(0, \infty)$ such that the following Riccati differential equation holds:*

$$\dot{P}(t) + \mathcal{A}^{\mathrm{T}}(t)P(t) + P(t)\mathcal{A}(t) + 2hP(t)A_1(t)A_1^{\mathrm{T}}(t)P(t) + \gamma I = 0. \tag{RDE}$$

Moreover, the solution $x(t, \phi)$ satisfies the condition

$$\|x(t, \phi)\| \leq \sqrt{\frac{p + \beta}{\beta}} e^{-\frac{\epsilon}{2(p+\beta)}t}\|\phi\|, \quad t \geq 0.$$

Proof. We use the Newton–Leibniz formula

$$x(t) - x(t - h(t)) = \int_{t-h(t)}^{t} \dot{x}(s)\mathrm{d}s,$$

and the system (2.1) leads to a new system

$$\begin{aligned}
&\dot{x}(t) = [A(t) + A_1(t)]x(t) - A_1(t)\int_{t-h(t)}^{t} A(s)x(s)\mathrm{d}s - A_1(t)\int_{t-h(t)}^{t} A_1(s)x(s - h(s))\mathrm{d}s,\\
&x(t) = \psi(t), \quad t \in [-2h, 0].
\end{aligned} \tag{3.1}$$

Note that the system (3.1) requires an initial function $\psi(t)$ on $[-2h, 0]$: $\psi(s) = \phi(s + h(0)), -h - h(0) \leq s \leq -h(0), \psi(s) = x(t + s), -h(0) \leq s \leq 0$, and $A(t) = A(0), A_1(t) = A_1(0), B(t) = B(0), t \in [-h, 0]$, and as shown in [9], it is a special case of the system (2.1) such that the stability property of the system (3.1) will ensure the stability property of the system (2.1). Therefore, we will consider the stability of the system (3.1) in order to ascertain the stability of system (2.1). For the system of (3.1), we consider the following Lyapunov–Krasovskii functional:

$$V(t, x) = \langle P(t)x, x\rangle + \beta\|x\|^2,$$

where $P(t)$ is a solution of (RDE). It is easy to see that

$$\beta\|x\|^2 \leq V(t, x) \leq (p + \beta)\|x\|^2, \quad \forall t \in R^+, x \in R^n. \tag{3.2}$$

The time derivative of $V(t, x)$ along the trajectory of the system (3.1) is given by

$$\begin{aligned}
\dot{V}(t, x(t)) = {} & \langle(\dot{P} + \overline{A}^T P_\beta + P_\beta\overline{A})x(t), x(t)\rangle - 2\langle P_\beta(t)A_1(t)\int_{t-h(t)}^{t} A(s)x(s)\mathrm{d}s, x(t)\rangle\\
& - 2\langle P_\beta(t)A_1(t)\int_{t-h(t)}^{t} A_1(s)x(s - h(s))\mathrm{d}s, x(t)\rangle.
\end{aligned}$$

For the chosen number $\lambda > 0$, we have $\lambda a \leq \beta, \lambda a_1 \leq \beta$; we get

$$\lambda\langle A(t)A^{\mathrm{T}}(t)x, x\rangle \leq \langle P_\beta(t)x, x\rangle, \qquad \lambda\langle A_1(t)A_1^{\mathrm{T}}(t)x, x\rangle \leq \langle P_\beta(t)x, x\rangle, \quad \forall t \geq 0, x \in R^n.$$

Therefore, the following estimates hold by applying Proposition 2.1 with $W = I$:

$$\begin{aligned}
-2\langle P_\beta(t)A_1(t)\int_{t-h(t)}^{t} A(s)x(s)\mathrm{d}s, x(t)\rangle &= \int_{t-h(t)}^{t} -2\langle P_\beta(t)A_1(t)A(s)x(s), x(t)\rangle\mathrm{d}s\\
&= \int_{t-h(t)}^{t} -2\langle A(s)x(s), A_1^{\mathrm{T}}(t)P_\beta(t)x(t)\rangle\mathrm{d}s\\
&\leq \int_{t-h(t)}^{t} \langle P_\beta(t)A_1(t)A_1^{\mathrm{T}}(t)P_\beta(t)x(t), x(t)\rangle\mathrm{d}s + \int_{t-h(t)}^{t} \langle A(s)A^{\mathrm{T}}(s)x(s), x(s)\rangle\mathrm{d}s\\
&\leq h\langle P_\beta(t)A_1(t)A_1^{\mathrm{T}}(t)P_\beta(t)x(t), x(t)\rangle + \lambda^{-1}\int_{t-h(t)}^{t} \langle P_\beta(s)x(s), x(s)\rangle\mathrm{d}s\\
&\leq h\langle P_\beta(t)A_1(t)A_1^{\mathrm{T}}(t)P_\beta(t)x(t), x(t)\rangle + \lambda^{-1}\int_{-h(t)}^{0} \langle P_\beta(t + s)x(t + s), x(t + s)\rangle\mathrm{d}s.
\end{aligned}$$

Similarly,

$$\begin{aligned}
-2\langle P_\beta(t)A_1(t)\int_{t-h(t)}^{t} A_1(s)x(s - h(s))\mathrm{d}s, x(t)\rangle &\leq h\langle P_\beta(t)A_1(t)A_1^{\mathrm{T}}(t)P_\beta(t)x(t), x(t)\rangle\\
&+ \lambda^{-1}\int_{-h(t)}^{0} \langle P_\beta(t + s - h(t + s))x(s - h(s)), x(s - h(s))\rangle\mathrm{d}s
\end{aligned}$$

Since $\langle P_\beta(t)x(t), x(t)\rangle = V(t, x(t))$, in the light of the Razumikhin theorem, we assume that for any real number $q > 1$ such that

$$V(t+s, x(t+s)) < qV(t, x(t)), \quad \forall s \in [-2h, 0], \forall t \geq 0$$

and using the condition (3.2), it is easy to obtain

$$\begin{aligned}\dot V(t, x(t)) &\leq \langle(\dot P + \overline{A}^T P_\beta + P_\beta \overline{A})x(t), x(t)\rangle + 2h\langle P_\beta A_1 A_1^{\mathrm T} P_\beta x(t), x(t)\rangle + 2hq\lambda^{-1}\langle P_\beta(t)x(t), x(t)\rangle \\ &\leq \langle(\dot P + \overline{A}^T P_\beta + P_\beta \overline{A} + 2hP_\beta A_1 A_1^{\mathrm T} P_\beta + 2hq\lambda^{-1} I)x(t), x(t)\rangle. \end{aligned} \tag{3.3}$$

Now taking $q \to 1^+$, (3.3) leads to

$$\dot V(t, x(t)) \leq \langle(\dot P + \overline{A}^T P_\beta + P_\beta \overline{A} + 2hP_\beta A_1 A_1^{\mathrm T} P_\beta + 2h\lambda^{-1} I)x(t), x(t)\rangle.$$

Therefore,

$$\dot V(t, x(t))\langle(\dot P + \mathcal{A}^T P + P\mathcal{A} + 2hPA_1^{\mathrm T} A_1 P + \gamma I)x(t), x(t)\rangle.$$

Since $P(t)$ is the solution of (RDE), we have

$$\dot V(t, x(t)) \leq -\epsilon\|x(t)\|^2, \quad \forall t \geq 0, \tag{3.4}$$

which, by the Razumikhin stability theorem, Proposition 2.2, implies the asymptotic stability of the system (3.1). To find the exponential factor of the solution, integrating both sides of the inequality, due to of (3.4), $\dot V(t, x(t)) \leq 0$ in t and using the condition (3.2), we have

$$\beta\|x(t, \phi)\| \leq V(t, x(t)) \leq V(0, x(0))\mathrm{e}^{-\frac{\epsilon}{p+\beta}t}$$

and hence

$$\|x(t, \phi)\| \leq \sqrt{\frac{p+\beta}{\beta}}\|\phi\|\mathrm{e}^{-\frac{\epsilon}{2(p+\beta)}t}, \quad \forall t \geq 0.$$

This completes the proof of the theorem. □

Remark 3.1. Note that from the proof of Theorem 3.1, the condition (RDE) can be relaxed via the following matrix inequality:

$$\dot P(t) + \mathcal{A}^{\mathrm T}(t)P(t) + P(t)\mathcal{A}(t) + 2hP(t)A_1(t)A_1^{\mathrm T}(t)P(t) + \gamma I \leq 0.$$

As an application, we apply the result obtained to the exponential stability of linear uncertain systems with time-varying delay considered in [11,12]:

$$\begin{aligned}&\dot x(t) = (A + H\Delta(t)E)x(t) + (A_1 + H\Delta_1(t)E_1)x(t - h(t)), \quad t \geq 0, \\ &x(t) = \phi(t), \quad t \in [-h, 0],\end{aligned} \tag{3.5}$$

where $0 \leq h(t) \leq h, A, A_1, H, E, E_1$ are constant matrices of appropriate dimensions and $\Delta(t), \Delta_1(t)$ are unknown time-varying uncertain matrices that satisfy

$$\Delta^{\mathrm T}(t)\Delta(t) \leq I, \quad \Delta_1^{\mathrm T}(t)\Delta_1(t) \leq I.$$

We have the following corollary.

Corollary 3.1. *The system (3.5) is exponentially stable if there exist a symmetric positive definite matrix X and positive numbers $\beta, \lambda, \epsilon_i, i = 1, 2, 3, 4$, such that $\lambda^{-1}\beta \geq \max\{a, a_1\}$, $\epsilon_4 I - E_1 E_1^{\mathrm T} > 0$ and the following LMI hold:*

$$\begin{pmatrix} \Omega & \overline{\gamma}X & XE^{\mathrm T} & XE_1^{\mathrm T} & 2\sqrt{h}A_1E_1^{\mathrm T} \\ \overline{\gamma}X & -\overline{\gamma}I & 0 & 0 & 0 \\ EX & 0 & -\epsilon_2 I & 0 & 0 \\ E_1X & 0 & 0 & -\epsilon_3 I & 0 \\ 2\sqrt{h}E_1A_1^{\mathrm T} & 0 & 0 & 0 & -\left(\epsilon_4 I - E_1E_1^{\mathrm T}\right) \end{pmatrix} \leq 0, \tag{3.6}$$

where

$$\overline{A} = A + A_1; \quad \overline{\gamma} = 2\beta\mu(\overline{A}) + 4h\beta^2 a_1 + 2h\lambda^{-1} + \epsilon_1;$$

$$\Omega = X\left(\overline{A} + 2h\lambda^{-1}I\right)^{\mathrm T} + \left(\overline{A} + 2h\lambda^{-1}I\right)X + \left(\epsilon_2 + \epsilon_3 + 4h\epsilon_4\right)HH^{\mathrm T} + 4hA_1A_1^{\mathrm T}.$$

Moreover, the solution $x(t,\phi)$ *of the system* (3.5) *satisfies*

$$\|x(t,\phi)\| \le N e^{-\sigma t}\|\phi\|, \quad t \ge 0,$$

where $N = \sqrt{\frac{\lambda_{\min}^{-1}(X)+\beta}{\beta}}, \quad \sigma = \frac{\epsilon_1}{2(\lambda_{\min}^{-1}(X)+\beta)}$.

Proof. Define $A(t) = A + H\Delta(t)E, A_1(t) = A_1 + H\Delta_1(t)E_1$ and $P = X^{-1}$; then we have

$$\begin{aligned}\mathcal{A}^{\mathrm{T}}(t)P + P\mathcal{A}(t) &= P(\bar{A} + 2h\lambda^{-1}I) + (\bar{A} + 2h\lambda^{-1}I)^{\mathrm{T}}P + PH(\Delta(t)E + \Delta_1(t)E_1) + (\Delta(t)E + \Delta_1(t)E_1)^{\mathrm{T}}H^{\mathrm{T}}P \\ &\quad + 2h\beta(PA_1(t)A_1^{\mathrm{T}}(t) + A_1(t)A_1^{\mathrm{T}}(t)P). \end{aligned} \tag{3.7}$$

Applying Proposition 2.3 implies

$$\begin{aligned} &PH\Delta(t)E + E^{\mathrm{T}}\Delta^{\mathrm{T}}(t)H^{\mathrm{T}}P \le \epsilon_2 PHH^{\mathrm{T}}P + \epsilon_2^{-1}E^{\mathrm{T}}E; \\ &PH\Delta_1(t)E_1 + E_1^{\mathrm{T}}\Delta_1^{\mathrm{T}}(t)H^{\mathrm{T}}P \le \epsilon_3 PHH^{\mathrm{T}}P + \epsilon_3^{-1}E_1^{\mathrm{T}}E_1; \\ &2h\beta(A_1(t)A_1^{\mathrm{T}}(t)P + PA_1(t)A_1^{\mathrm{T}}(t)) \le 2hPA_1(t)A_1^{\mathrm{T}}(t)P + 2h\beta^2A_1(t)A_1^{\mathrm{T}}(t) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\;\; \le 2hPA_1(t)A_1^{\mathrm{T}}(t)P + 2h\beta^2a_1I. \end{aligned}$$

Therefore, from (3.7) we obtain

$$\begin{aligned}\mathcal{A}^{\mathrm{T}}(t)P + P\mathcal{A}(t) &\le P(\bar{A} + 2h\lambda^{-1}I) + (\bar{A} + 2h\lambda^{-1}I)^{\mathrm{T}}P + \epsilon_2 PHH^{\mathrm{T}}P + \epsilon_3 PHH^{\mathrm{T}}P + \epsilon_2^{-1}E^{\mathrm{T}}E + \epsilon_3^{-1}E_1^{\mathrm{T}}E_1 \\ &\quad + 2hPA_1(t)A_1^{\mathrm{T}}(t)P + 2h\beta^2a_1I. \end{aligned} \tag{3.8}$$

Applying Proposition 2.3 again we have

$$A_1(t)A_1^{\mathrm{T}}(t) \le A_1A_1^{\mathrm{T}} + A_1E_1^{\mathrm{T}}(\epsilon_4I - E_1E_1^{\mathrm{T}})^{-1}E_1A_1^{\mathrm{T}} + \epsilon_4HH^{\mathrm{T}}.$$

Hence, taking (3.8) into account, we have

$$\begin{aligned}&\mathcal{A}^{\mathrm{T}}(t)P + P\mathcal{A}(t) + 2hPA_1(t)A_1^{\mathrm{T}}(t)P + \gamma I \le P(\bar{A} + 2h\lambda^{-1}I) + (\bar{A} + 2h\lambda^{-1}I)^{\mathrm{T}}P + 4hPA_1A_1^{\mathrm{T}}P \\ &\quad + (\epsilon_2 + \epsilon_3 + 4h\epsilon_4)PHH^{\mathrm{T}}P + \epsilon_2^{-1}E^{\mathrm{T}}E + \epsilon_3^{-1}E_1^{\mathrm{T}}E_1 + 4hPA_1E_1^{\mathrm{T}}(\epsilon_4I - E_1E_1^{\mathrm{T}})^{-1}E_1A_1^{\mathrm{T}}P + \bar{\gamma}I. \end{aligned} \tag{3.9}$$

By pre-multiplying and post-multiplying the right hand side of (3.9) with X and using Schur complement theorem, it follows that

$$\mathcal{A}^{\mathrm{T}}(t)P + P\mathcal{A}(t) + 2hPA_1(t)A_1^{\mathrm{T}}(t)P + \gamma I \le 0.$$

By Theorem 3.1 and Remark 3.1, the system (3.5) is exponentially stable, which completes the proof of the corollary. □

Remark 3.2. Note that the stability conditions were proposed in [2,5–7] under the assumption on the differentiability of the delay function. Moreover, the use of the Newton–Leibniz formula in our method allows us to get less conservative stability conditions. In the sequel, we shall show that the uncertain linear system (3.5) with fast-varying delay is exponentially stable, while the nominal undelayed system is unstable.

Example 3.1. Consider uncertain linear system (3.5) where

$$h(t) = \begin{cases} 0.03\sin t, & \textit{if} \quad t \in I = [2k\pi, (2k+1)\pi], k = 0, 1, 2, \ldots \\ 0 \textit{if} \quad t \in R^{+} \setminus I, \end{cases}$$

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -4 & 1 \\ 0 & -3 \end{pmatrix}, \quad H = I, E = 0.2I, E_1 = 0.$$

It is easy to verify that the nominally undelayed system

$$\dot{x}(t) = Ax(t)$$

is unstable and the delay function $h(t)$ is non-differentiable and thus the results of [2,5–7] are not applicable. However, for $\lambda = 0.25$, $\beta = 4$, $\epsilon_1 = 0.1$, $\epsilon_2 = \epsilon_3 = 0.5$, $\epsilon_4 = 1.04$, then all conditions in Corollary 3.1 and LMI (3.6) are satisfied with

$$X = \begin{pmatrix} 0.8355 & -0.0977 \\ -0.0977 & 0.9549 \end{pmatrix}.$$

By Corollary 3.1, the system is exponentially stable and the solution $x(t,\phi)$ satisfies

$$\|x(t,\phi)\| \le 1.149e^{-0.0095t}\|\phi\|, \quad t \ge 0.$$

Remark 3.3. The sufficient stability condition in Theorem 3.1 is given in terms of the solution of RDEs. Although the problem of solving of RDEs is in general still not easily addressed, various effective approaches for finding the solutions of RDEs can be found in [13,14].

4. Conclusions

This work has been concerned with the problem of exponential stability for linear non-autonomous systems with time-varying delay. In the light of the Razumikhin stability theorem combined with the Newton–Leibniz formula, exponential stability conditions are derived in terms of the solution of Riccati differential equations without assumption on the differentiability of the time delay function.

Acknowledgments

This work was supported by the National Foundation for Science and Technology Development, Vietnam. The authors would like to thank the anonymous referees and Professor L. Pandolfi for valuable remarks and suggestions, which allowed us to improve the work.

References

[1] P.P. Civalleri, M. Gilli, L. Pandolfi, On stability of cellular neural networks with delay, IEEE Trans. Circuits Systems I: Fund. Theory Appl. 40 (1993) 157–165.
[2] O.M. Kwon, J.H. Park, S.M. Lee, Exponential stability for uncertain dynamic systems with time-varying delays: LMI optimization approach, J. Optim. Theory Appl. 137 (2008) 521–532.
[3] L. Pandolfi, Exponential stability and Lyapunov equation for a class of functional-differential equations, Boll. Un. Mat. Ital. B 16 (1979) 897–909.
[4] V.N. Phat, New stabilization criteria for linear time-varying systems with state delay and norm-bounded uncertainties, IEEE Trans. Aut. Contr. 47 (2002) 2095–2098.
[5] P. Niamsup, K. Mukdasai, V.N. Phat, Improved exponential stability for time-varying systems with nonlinear delayed perturbations, Appl. Math. Comput. 204 (2008) 490–495.
[6] V.N. Phat, P. Niamsup, Stability of linear time-varying delay systems and applications to control problems, J. Comput. Appl. Math. 194 (2006) 343–356.
[7] H. Yan, X. Huang, M. Wang, H. Zhang, New delay-dependent stability criteria of uncertain linear systems with multiple time-varying state delays, Chaos, Solitons Fractals 37 (2008) 157–165.
[8] X. Li, C.E. de Souza, Criteria for robust stability and stabilization of uncertain linear systems with state-delay, Automatica 33 (1997) 1657–1662.
[9] J.K. Hale, S.M. Verduyn Lunel, Introduction to Functional Differential Equations, Springer-Verlag, New York, 1993.
[10] Y. Wang, L. Xie, C.E de Souza, Robust control of a class of uncertain nonlinear systems, Systems Control Lett. 19 (1992) 139–149.
[11] E. Fridman, S.-I. Niculescu, On complete Lyapunov–Krasovskii functional techniques for uncertain systems with fast-varying delays, Internat. J. Robust Nonlinear Control 18 (2007) 364–374.
[12] E. Fridman, U. Shaked, Input–output approach to stability and L_2-gain analysis of systems with time-varying delays, Systems Control Lett. 55 (2006) 1041–1053.
[13] A.J. Laub, Schur techniques for solving Riccati equations, IEEE Trans. Automat. Control 24 (1979) 913–921.
[14] T. William, Riccati Differential Equations, Academic Press, New York, 1972.