MODULE 5

THE DISCRETE KALMAN FILTER

- Like the Wiener filter, the Kalman filter is a solution to the MMSE estimation problem.

- There are two main differences between the Kalman filter and the Wiener filter:

  1. For the Kalman filter, the problem is formulated in state space.
  2. Whereas the Wiener filter is the optimal LSI estimator, the Kalman filter is time varying in general.

- Thus, we begin by reviewing the relationship between our standard spectral representation of a stationary stochastic process and the "controllable canonical form" state space model.

- Consider a linear system

$$u(t) \longrightarrow \boxed{\text{system}} \longrightarrow y(t)$$

where u(t) and y(t) could be scalars or vectors.

→ Each element of u(t) is a white noise process with unit-magnitude power spectrum.

- A general state space model for the system is

$$\begin{aligned} \dot{x} &= Fx + Gu \\ y &= Bx \end{aligned} \qquad (*)$$

⟶

where:

F, G are matrices of appropriate size.

B is a matrix or a vector of appropriate size.

X is a vector of intermediate variables called "state variables" or "phase variables".

→ X is called the "state vector" of the system.

- The state space model is <u>not</u> unique. For any given system, there exist uncountably infinitely many models of the form $(*)$ on PAGE 5.1.

- For a system with a transfer function that is rational, a finite dimensional model of the form $(*)$ on PAGE 5.1 always exists.

- For the moment, let us assume that $u(t)$ and $y(t)$ are scalar processes and that the PSD $S_y(s)$ of $y(t)$ is known.

- We want to model $y(t)$ as the response of a linear system, where the system input is the unity power spectrum white noise $u(t)$.

  <u>Note</u>: this implies that $E[x(t)] = 0$.

- we begin by spectrally factorizing $S_y(s)$:

$$S_y(s) = S_y^+(s)\, S_y^-(s)$$

where

$S_y^+(s)$ has left half-plane poles and zeros only.

$S_y^-(s)$ has right half-plane poles and zeros only.

- Now, the shaping filter required to generate $y(t)$ with nontrivial correlation structure from the process $X(t)$ with trivial correlation structure is simply $S_y^+(s)$.

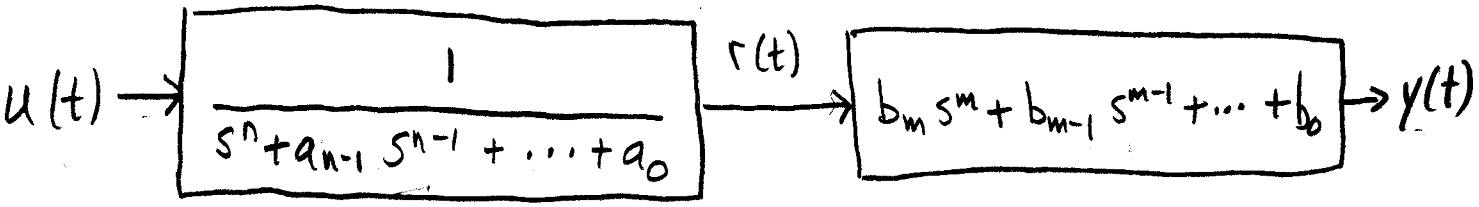
- Suppose $S_y^+(s) = \dfrac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_0}$, where $m = n-1$.

- We have then

$$u(t) \longrightarrow \boxed{\dfrac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_0}} \longrightarrow y(t)$$

- We first "break apart" the denominator and numerator of $S_y^+(s)$.

- Between the denominator and numerator, we introduce the intermediate process $r(t)$:


$u(t)$
$\frac{1}{s^n + a_{n-1}s^{n-1} + \dots + a_0}$
$r(t)$
$b_m s^m + b_{m-1}s^{m-1} + \dots + b_0$
$y(t)$

- To be concrete, letus assume that $n=4$, $b=3$.

- We next define the state variables according to:

$$\left.\begin{aligned} x_1 &= r(t) \\ x_2 &= \dot{r}(t) = \dot{x}_1 \\ x_3 &= \ddot{r}(t) = \dot{x}_2 \\ x_4 &= \dddot{r}(t) = \dot{x}_3 \end{aligned}\right\} (*)$$

- In the Laplace domain, the relationship between $U(s)$ and $R(s)$ is

$$U(s) = s^4 R(s) + a_3 s^3 R(s) + a_2 s^2 R(s) + a_1 s R(s) + a_0 R(s)$$

- So

$$u(t) = \ddddot{r}(t) + a_3 \dddot{r}(t) + a_2 \ddot{r}(t) + a_1 \dot{r}(t) + a_0 r(t)$$

$\longrightarrow$

- we have

$$\ddddot{r}(t) = -a_0 r(t) - a_1 \dot{r}(t) - a_2 \ddot{r}(t) - a_3 \dddot{r}(t) + u(t)$$

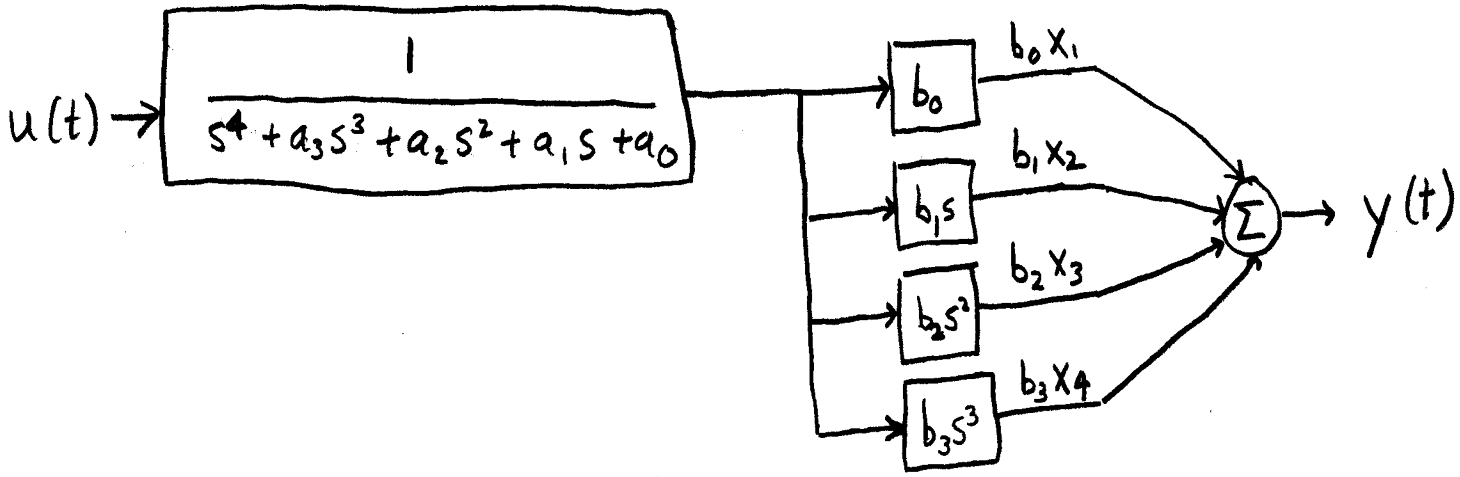
$$\dot{x}_4 = -a_0 x_1 - a_1 x_2 - a_2 x_3 - a_3 x_4 + u(t)$$

- Writing this last equation together with equations (*) on PAGE 5.4 using matrix notation, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t) \qquad (*)$$

$$\dot{x} = F \qquad x + G u$$

- Now, we redraw the block diagram on PAGE 5.4 as



$$y(t) = b_0 x_1 + b_1 x_2 + b_2 x_3 + b_3 x_4$$

- In matrix notation,

$$y(t) = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 & b_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad (*)$$

$$y \;=\; B \quad X$$

- Together, the state equation (*) on PAGE 5.5 and the output equation (*) above constitute the controllable canonical state variable model for the system.
- In general, this model is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & \\ 0 & 0 & 1 & & \\ & & & \ddots & \\ -a_0 & -a_1 & \cdots & & -a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} b_0 & b_1 & \cdots & b_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

- Carefully study examples 5.1 and 5.2 on pages 194-196 of the book.

- The fundamental requirement that determines whether a stochastic process $y(t)$ can be formulated as a state space model of the type we have just examined is this:

  Such a model exists if $y(t)$ can be related to a white noise through a linear differential equation.

- Several examples are given on pages 196-198 of the book. Study them carefully.

Discrete-Time State Space Model

- First, we look at the relationship between the controllable canonical form and the ARMA model.

- Let W_K be a white noise vector with known diagonal covariance matrix Q_K given by

$$E\left[W_K W_i^T\right] = Q_K \delta_{i-K} .$$

NOTE: In general, the diagonal matrix Q_K will be time varying.

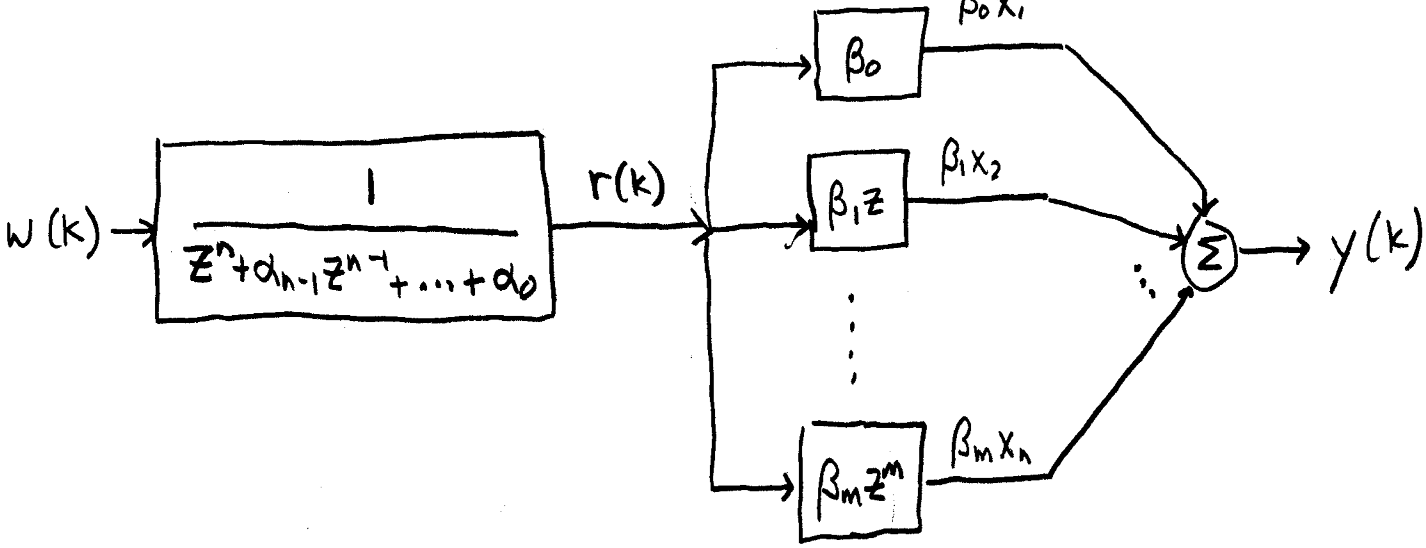
- assume for the moment that w_k is a scalar process and let the scalar process y_k be related to u_k through an LSI shaping filter with a rational transfer function.
- We want to construct the controllable canonical state space model for the process y_k.
- In the Kalman filtering literature, it is customary to write $w(k)$ and $y(k)$ for the discrete-time processes w_k and y_k.
- we have

$$w(k) \rightarrow \boxed{\frac{\beta_m z^m + \beta_{m-1} z^{m-1} + \beta_{m-2} z^{m-2} + \cdots + \beta_0}{z^n + \alpha_{n-1} z^{n-1} + \alpha_{n-2} z^{n-2} + \cdots + \alpha_0}} \rightarrow y(k),$$

where $m = n-1$.

- To be concrete, let us assume that $n=4$, so that $m=3$.
- As before, we pull apart the numerator and denominator of the transfer function.
- Between them, we define an intermediate stochastic process $r(k)$.

→



- Define the state variables according to (with $n=4$):

$$\left.\begin{aligned} x_1(k) &= r(k) \\ x_2(k) &= r(k+1) = x_1(k+1) \\ x_3(k) &= r(k+2) = x_2(k+1) \\ x_4(k) &= r(k+3) = x_3(k+1) \end{aligned}\right\} \quad (*)$$

- The relationship between $W(z)$ and $R(z)$ is

$$W(z) = \alpha_0 R(z) + \alpha_1 z R(z) + \alpha_2 z^2 R(z) + \alpha_3 z^3 R(z) + z^4 R(z)$$

$$w(k) = \alpha_0 r(k) + \alpha_1 r(k+1) + \alpha_2 r(k+2) + \alpha_3 r(k+3) + r(k+4)$$

$$r(k+4) = -\alpha_0 r(k) - \alpha_1 r(k+1) - \alpha_2 r(k+2) - \alpha_3 r(k+3) + w(k)$$

$$x_4(k+1) = -\alpha_0 x_1(k) - \alpha_1 x_2(k) - \alpha_2 x_3(k) - \alpha_3 x_4(k) + w(k) \quad (**)$$

- From the figure above, we have also that

$$y(k) = \beta_0 x_1(k) + \beta_1 x_2(k) + \beta_2 x_3(k) + \beta_3 x_4(k) \quad (***)$$

- Writing equations (*) and equation (***) on PAGE 5.9 together using matrix notation, we obtain the controllable canonical state variable form

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w(k)$$

$$y(k) = \begin{bmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix}$$

- OR,

$$X_{k+1} = \Phi_k X_k + W_k$$

$$Y_k = B_k X_k$$

where

X_k = state vector at time k.

Φ_k = state transition matrix for $X_k \rightarrow X_{k+1}$

B_k = output equation gain matrix

-From the figure on PAGE 5.8, we have

$$\frac{Y(z)}{W(z)} = \frac{\beta_m z^m + \beta_{m-1} z^{m-1} + \cdots + \beta_0}{z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_0}$$

$$Y(z)\left[z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_0\right] = W(z)\left[\beta_m z^m + \beta_{m-1} z^{m-1} + \cdots + \beta_0\right]$$

- Thus, the ARMA model relating $y(k)$ and $w(k)$ is:

$$y(k+n) + \alpha_{n-1} y(k+n-1) + \alpha_{n-2} y(k+n-2) + \cdots + \alpha_0 y(k)$$
$$= \beta_m w(k+m) + \beta_{m-1} w(k+m-1) + \beta_{m-2} w(k+m-2) + \beta_0 w(k)$$

-The corresponding controllable canonical state space model is:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & & & & \ddots \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + w(k)$$

$$y(k) = \left[\beta_0\ \beta_1\ \beta_2 \cdots \beta_n\right] \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix}$$

Sampling of a Continuous-Time System

- Sampling of the continuous-time output equation

$$y = Bx$$

is obvious, so we concentrate on the state equation.

- The continuous-time state equation is

$$\dot{x} = Fx + Gu$$

- We sample this equation at discrete times $t_0, t_1, t_2, \cdots t_k, \cdots$ to obtain

$$x(t_{k+1}) = \underbrace{\Phi(t_{k+1}, t_k)}_{\Phi_k} x(t_k) + \underbrace{\int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \tau) G(\tau) u(\tau)\, d\tau}_{w_k}$$

$$\Rightarrow \quad x_{k+1} = \Phi_k x_k + w_k \qquad (*)$$

- It is necessary to solve for Φ_k and Q_k from the specification of the continuous-time system.
- If the matrix F is not time varying and if the sampling instants are equally spaced by a time Δt, then

$$\Phi_k = \Big[\mathcal{L}^{-1}\{(sI - F)^{-1}\}\Big]_{t=\Delta t}.$$

- The input covariance matrix is given by

$$E\left[w_k w_i^T\right] = Q_k\,\delta(k-i)\ ,\ \text{where}$$

$$Q_k = E\left[\int_{t_k}^{t_{k+1}} \Phi(t_{k+1},\theta)G(\theta)u(\theta)d\theta \left[\int_{t_k}^{t_{k+1}} \Phi(t_{k+1},\psi)G(\psi)u(\psi)d\psi\right]^T\right]$$

$$= \int_{t_k}^{t_{k+1}}\int_{t_k}^{t_{k+1}} \Phi(t_{k+1},\theta)G(\theta)E\left[u(\theta)u^T(\psi)\right]G^T(\psi)\Phi^T(t_{k+1},\psi)\,d\psi\, d\theta .$$

NOTE: Since $u(t)$ is a white noise,

$E\left[u(\theta)u^T(\psi)\right]$ is a diagonal matrix.

→ The diagonal entries are Dirac deltas.

- Study example 5.3 on page 200 of the book.
- A numerical method for finding Q_k is given on page 204 of book.

NOTE: if F is time varying, Φ_k must be obtained by solving the differential equation

$$\dot{\Phi}(t,t_k) = F(t)\Phi(t,t_k)\ ;\ \Phi(t_k,t_k) = I.$$

- If $F(t)$ does not vary from t_k to t_{k+1}, then

$$\Phi_k = e^{F(t_{k+1}-t_k)} \quad \text{(Matrix exponential)}.$$

A Development of the Discrete Kalman Filter

- In a nutshell:
  - There is a discrete linear system.
  - The input is white noise.
  - Your observations are the system output plus a white noise called the "measurement noise."

  → The system input noise and the measurement noise are uncorrelated with each other.
  - You know:
    - The state space model for the system.
    - The 2nd order statistics of the input noise.
    - The 2nd order statistics of the measurement noise.
- The problem:
  - Given the noisy observations of the output, find MMSE estimates of the system state vector.

NOTE: if you can get MMSE estimates of the state vector, then it is trivial to turn these into MMSE estimates of an arbitrary affine function of the state vector... which is a statement of the general Kalman filtering problem.

NOTE: The system output itself is an affine function of the state vector. Thus, one of the most straightforward Kalman filtering problems is:

> given a noisy observation of the ouput, find MMSE estimates of the output.

- The system state space model:

$$\underset{\underset{[\;]\,(n\times 1)}{\uparrow}}{x_{k+1}} = \underset{\underset{[\quad]\,(n\times n)}{\uparrow}}{\phi_k}\,\underset{\underset{[\;]\,(n\times 1)}{\uparrow}}{x_k} + \underset{\underset{[\;]\,(n\times 1)}{\uparrow}}{w_k}$$

$$\underset{\underset{[\;]\,(m\times 1)}{\uparrow}}{y_k} = \underset{\underset{[\quad]\,(m\times n)}{\uparrow}}{H_k}\,\underset{\underset{[\;]\,(n\times 1)}{\nwarrow}}{x_k}$$

- The $m \times 1$ observation vector z_k is the system output y_k plus the $m \times 1$ measurement noise v_k:

$$z_k = H_k x_k + v_k$$

- The covariance structures of the input noise and measurement noise are known:

$$E\left[w_k w_i^T\right] = \begin{cases} Q_k, & i=k \\ 0, & i \neq k \end{cases} = Q_k \delta(i-k)$$

$$E\left[v_k v_i^T\right] = \begin{cases} R_k, & i=k \\ 0, & i \neq k \end{cases} = R_k \delta(i-k)$$

- The input noise and measurement noise are uncorrelated:

$$E\left[w_k v_i^T\right] = 0. \quad (\forall\, i, k)$$

All together:

$$x_{k+1} = \phi_k x_k + w_k \qquad (5.5.1)$$

$$z_k = H_k x_k + v_k \qquad (5.5.2)$$

$$E[w_k w_i^T] = Q_k \delta(i-k) \qquad (5.5.3)$$

$$E[v_k v_i^T] = R_k \delta(i-k) \qquad (5.5.4)$$

$$E[w_k v_i^T] = 0 \qquad (5.5.5)$$

- What makes the Kalman filter different from anything that came before is the recursive way it is formulated.

  → In this regard, it is kind of like a mathematical proof by induction.

- Assume that somehow we have obtained a prediction for the state vector at time k,

  AND

  that this estimate is based on the first k-1 observations.

⇒ In other words, assume that we have an estimate of x_k given $\ldots, z_{k-3}, z_{k-2}, z_{k-1}$.

→ This is often written $\hat{x}_{k|k-1}$.

→ But in the book it is written $\hat{x}_k^-$.

NOTE: $\hat{x}_k^-$ is an "a priori estimate" or "prior" of x_k, the true state vector at time k.

NOTE: Theassumption that we "somehow" have this prediction $\hat{x}_k^-$ is not a problem: in the worst case, we can just set $\hat{x}_k^- = E[x_k]$, which is usually zero...

NOTE: $E[x_k]$ is computable (solvable). If it turns out to be nonzero, then it is customary to subtract it out of the model. This leads to a modified system model where $E[x_k]=0$.

NOTE: $\hat{x}_k^-$ might be a good estimate or a poor estimate... this doesn't matter.

→ What does matter is that we need $\hat{x}_k^-$ to be unbiased. That is, we need it to be true that

$$E[\hat{x}_k^-] = E[x_k].$$

- So the error in the prediction $\hat{x}_k^-$ is

$$e_k^- = \underset{\substack{\uparrow \\ \text{true}}}{x_k} - \underset{\substack{\uparrow \\ \text{predicted}}}{\hat{x}_k^-} \qquad (5.5.6)$$

with mean
$$E[e_k^-] = E[x_k - \hat{x}_k^-]$$
$$= E[x_k] - E[\hat{x}_k^-] = 0.$$

- e_k is the "predicted state vector error."
- The predicted state vector error covariance matrix is:

$$P_k^- = E[e_k^- e_k^{-T}] = E[(x_k - \hat{x}_k^-)(x_k - \hat{x}_k^-)^T] \qquad (5.5.7)$$

- Now, given $\hat{x}_k^-$ and the observation z_k, we want to "refine" the prediction $\hat{x}_k^-$ by using the new information in z_k in an optimal way.

- This will give us an "a posteriori" estimate of x_k given $\underline{\underline{k}}$ observations (a "filtered" estimate)

  → often, this is written $\hat{x}_{k|k}$.

  → In the book, it is written $\hat{x}_k$.

- The strategy is to make the new & improved estimate $\hat{x}_k$ a linear combination, or "blending" of the old estimate $\hat{x}_k^-$ and the new observation z_k.

- The "blending coefficients" are going to be written in a weird way that will prove useful a little bit later:

$$\hat{x}_k = \underbrace{(I - K_k H_k)}_{\substack{\text{blending factor} \\ \text{for the old a priori} \\ \text{estimate } \hat{x}_k^-}} \hat{x}_k^- + \underbrace{K_k}_{\substack{\text{blending factor} \\ \text{for the new} \\ \text{observation } z_k\text{.}}} z_k$$

NOTE: This can be re-written as

$$\hat{x}_K = \hat{x}_K^- + K_K(z_K - H_K \hat{x}_K^-) \quad (5.5.8)$$

- We will seek to find a value for K_K that makes $\hat{x}_K$ optimal in the MMSE sense.

NOTE: in (5.5.8), the term $z_K - H_K \hat{x}_K^-$ may be considered as a "residual"... because, according to the measurement equation (5.5.2) on PAGE 5.16, $H_K \hat{x}_K^-$ is in fact an a priori estimate for z_K.

The error in this estimate, $z_K - H_K \hat{x}_K^-$, has two parts:

- The part due to the measurement noise v_K (undesirable).
- The part due to new information that is present in z_K but was not present in ... z_{K-3}, z_{K-2}, z_{K-1} (desirable!)

- The error in the a posteriori filtered state vector estimate $\hat{x}_K$ is

$$e_K = \underset{\substack{\uparrow \\ \text{True}}}{x_K} - \underset{\substack{\nwarrow \\ \text{estimated}}}{\hat{x}_K}$$

- The filtered state vector error covariance matrix is

$$
\begin{aligned}
P_K &= E\left[e_K e_K^T\right] \\
&= E\left[(x_K - \hat{x}_K)(x_K - \hat{x}_K)^T\right] \\
\text{use (5.5.8)} \quad &= E\left\{\left[x_K - \hat{x}_K^- - K_K(z_K - H_K\hat{x}_K^-)\right]\left[\text{same}\right]^T\right\} \\
\text{use (5.5.2)} \quad &= E\left\{\left[(x_K - \hat{x}_K^-) - K_K(H_K x_K + V_K - H_K\hat{x}_K^-)\right]\left[\text{same}\right]^T\right\} \\
&= E\left\{\left[(x_K - \hat{x}_K^-) - K_K H_K (x_K - \hat{x}_K^-) - K_K V_K\right]\left[\text{same}\right]^T\right\} \\
&= E\left\{\left[(I - K_K H_K)(x_K - \hat{x}_K^-) - K_K V_K\right]\right. \\
&\qquad \left. \times \left[(x_K - \hat{x}_K^-)^T(I - K_K H_K)^T - V_K^T K_K^T\right]\right\}
\end{aligned}
$$

⟶

$$
\begin{aligned}
P_k &= \ldots \\
&= E\{ (I-K_kH_k)(x_k-\hat{x}_k^-)(x_k-\hat{x}_k^-)^T(I-K_kH_k)^T \\
&\qquad - (I-K_kH_k)(x_k-\hat{x}_k^-)v_k^T K_k^T \\
&\qquad - K_k v_k (x_k-\hat{x}_k^-)^T(I-K_kH_k)^T + K_k v_k v_k^T K_k^T \} \\
&= (I-K_kH_k)E\{(x_k-\hat{x}_k^-)(x_k-\hat{x}_k^-)^T\}(I-K_kH_k)^T \\
&\qquad - (I-K_kH_k)\,E\{\underbrace{(x_k-\hat{x}_k^-)\,v_k^T}_{\text{zero}}\}K_k^T \\
&\qquad - K_k E\{\underbrace{v_k(x_k-\hat{x}_k^-)^T}_{\text{zero}}\}(I-K_kH_k)^T \\
&\qquad + K_k E\{v_k v_k^T\}K_k^T \\
&= (I-K_kH_k)P_k^-(I-K_kH_k)^T + K_k R_k K_k^T
\end{aligned}
$$

$$\boxed{P_k = (I-K_kH_k)P_k^-(I-K_kH_k)^T + K_k R_k K_k^T \quad (5.5.11)}$$

- This is true for any choice of K_k. The question is: which choice is optimal?

- Recall:
  - $\hat{x}_k$ is the filtered, or "updated" state vector estimate obtained using k observations.
  - The error in $\hat{x}_k$ is $e_k = x_k - \hat{x}_k$.
  - Assuming that $\hat{x}_k$ is an unbiased estimate of x_k, the error the filtered state vector error covariance matrix is

$$P_k = E\left[e_k e_k^T\right] = (5.5.11) \ldots \text{ which depends on } K_k.$$

- The main diagonal entries of P_k are the variances of the estimation errors in each component of $\hat{x}_k$.
- So what we want is a choice for the gain or "blending factor" K_k that will <u>minimize</u> the sum of the main diagonal entries in P_k.
- So we will take the derivative of the trace $\mathrm{Tr}\, P_k$ with respect to K_k, set this equal to zero, and solve for the optimal K_k.

NOTE: Tr A is a scalar... it is the sum of the main diagonal elements of the matrix A.

Note: The derivative of a scalar θ with respect to an $m \times n$ matrix $A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$

is given by

$$\frac{d\theta}{dA} = \begin{bmatrix} \frac{d\theta}{da_{1,1}} & \frac{d\theta}{da_{1,2}} & \cdots & \frac{d\theta}{da_{1,n}} \\ \frac{d\theta}{da_{2,1}} & \frac{d\theta}{da_{2,2}} & \cdots & \frac{d\theta}{da_{2,n}} \\ \vdots & \vdots & \ddots & \\ \frac{d\theta}{da_{m,1}} & \frac{d\theta}{da_{m,2}} & \cdots & \frac{d\theta}{da_{m,n}} \end{bmatrix}$$

FACTS From Matrix Theory:

- if A and B are conformable matrices and AB is a square matrix, then

$$\frac{d(\mathrm{Tr}AB)}{dA} = B^T$$

Matrix Theory Facts...

- if A and C are conformable matrices and C is a symmetric matrix, then

$$\frac{d(\mathrm{Tr}\; ACA^T)}{dA} = 2AC$$

- Now let us return to the expression (5.5.11) for P_k on PAGE 5.22, and rewrite it as

$$\begin{aligned}
P_k &= (I - K_k H_k) P_k^- (I - K_k H_k)^T + K_k R_k K_k^T \qquad (5.5.11) \\
&= (P_k^- - K_k H_k P_k^-)(I - H_k^T K_k^T) + K_k R_k K_k^T \\
&= P_k^- - P_k^- H_k^T K_k^T - K_k H_k P_k^- + K_k H_k P_k^- H_k^T K_k^T \\
&\qquad + K_k R_k K_k^T \\
&= P_k^- - K_k H_k P_k^- - P_k^- H_k^T K_k^T + K_k (H_k P_k^- H_k^T + R_k) K_k^T
\end{aligned}$$

- Now differentiate $\mathrm{Tr}\, P_k$ with respect to K_k :

$$\frac{d(\mathrm{Tr}\, P_k)}{dK_k} = \overbrace{\frac{d}{dK_k}\mathrm{Tr}\, P_k^-}^{\text{zero}} \overbrace{- \frac{d}{dK_k}\mathrm{Tr}\, K_k H_k P_k^- - \frac{d}{dK_k}\mathrm{Tr}\, P_k^- H_k^T K_k^T}^{\text{(same)}}$$

$$+ \frac{d}{dK_k}\mathrm{Tr}\, K_k (H_k P_k^- H_k^T + R_k) K_k^T$$

$$= -(H_k P_k^-)^T - \underbrace{H_k^T K_k^T}_{\substack{\| \\ (H_k K_k)^T}} + 2K_k(H_k P_k^- H_k^T + R_k)$$

$$= -2(H_k P_k^-)^T + 2K_k(H_k P_k^- H_k^T + R_k) \quad (5.5.16)$$

- So the optimal "kalman gain" K_k is found by solving

$$-2(H_k P_k^-)^T + 2K_k(H_k P_k^- H_k^T + R_k) = 0$$

$$(H_k P_k^-)^T = K_k(H_k P_k^- H_k^T + R_k)$$

$$\underline{\underline{K_k = (H_k P_k^-)^T (H_k P_k^- H_k^T + R_k)^{-1}}} \quad (5.5.17)$$

Review:

- we started with the state model and observation equation:

$$x_{k+1} = \phi_k x_k + w_k \qquad (5.5.1)$$

$$z_k = H_k x_k + v_k \qquad (5.5.2)$$

- we assumed that we were given an initial a priori state vector estimate $\hat{x}_k^-$ with error

$$e_k^- = x_k - \hat{x}_k^- \qquad (5.5.6)$$

and predicted state vector error covariance matrix

$$P_k^- = E[e_k^- e_k^{-T}] = E[(x_k - \hat{x}_k^-)(x_k - \hat{x}_k^-)^T]. \qquad (5.5.7)$$

- we assumed an update equation to revise the prediction $\hat{x}_k^-$ using the observation z_k:

$$\hat{x}_k = \hat{x}_k^- + K_k \underbrace{(z_k - H_k \hat{x}_k^-)}_{\text{residual... new information.}} \qquad (5.5.8)$$

- giving a smoothed state vector error covariance matrix of

$$P_k = (I - K_k H_k) P_k^- (I - K_k H_k)^T + K_k R_k K_k^T \qquad \begin{matrix}(5.5.11)\\(5.5.18)\end{matrix}$$

- We found the optimal MMSE Kalman gain vector K_k by solving

$$\frac{\partial \, \mathrm{Tr}(P_k)}{\partial K_k} = 0$$

for K_k, giving

$$K_k = P_k^- H_k^T \left(H_k P_k^- H_k^T + R_k \right)^{-1} \quad (5.5.17)$$

- Now, we use this to simplify the general expression (5.5.11) for P_k in the special case that K_k is the optimal MMSE gain solution:

$$\begin{aligned} P_k &= (I - K_k H_k) P_k^- (I - K_k H_k)^T + K_k R_k K_k^T \quad (5.5.11)(5.5.18) \\ &= \left(P_k^- - K_k H_k P_k^-\right)\left(I - H_k^T K_k^T\right) + K_k R_k K_k^T \\ &= P_k^- - K_k H_k P_k^- - P_k^- H_k^T K_k^T + K_k H_k P_k^- H_k^T K_k^T \\ &\qquad + K_k R_k K_k^T \\ &= P_k^- - K_k H_k P_k^- - P_k^- H_k^T K_k^T + K_k \left(H_k P_k^- H_k^T + R_k\right) K_k^T \end{aligned}$$

- Substituting in (5.5.17) for K_k yields: (5.5.19)

$$P_k = (I - K_k H_k) P_k^- \quad (5.5.22)$$

- We now have four different equations for P_k:

$$P_k = (I - K_k H_k) P_k^- (I - K_k H_k)^T + K_k R_k K_k^T \qquad (5.5.11) \quad (5.5.18)$$

→ good for <u>any</u> choice of K_k

$$P_k = P_k^- - P_k^- H_k^T (H_k P_k^- H_k^T + R_k)^{-1} H_k P_k^- \qquad (5.5.20)$$

$$P_k = P_k^- - K_k (H_k P_k^- H_k^T + R_k) K_k^T \qquad (5.5.21)$$

$$P_k = (I - K_k H_k) P_k^- \qquad (5.5.22)$$

→ valid <u>only</u> for the optimal K_k.

→ intermediate results obtained in going from (5.5.11/18) to (5.5.22).

- Any of these four expressions can be used in practice to obtain P_k.
  - → In various practical situations with finite percision arithmetic, one or the other of these may give superior numerical performance and/or convergence behavior.
  - → Important, since it is often necessory to compute P_k "on the fly" as the kalman filter runs.

- For now, we will assume that it's sufficient to calculate P_k using

$$P_k = (I - K_k H_k) P_k^- \qquad (5.5.22)$$

- Given that we now have $\hat{x}_k^-$ and $\hat{x}_k$, we need to "age" the state vector estimate by predicting it ahead to time k+1.

→ Since the input w_k is assumed zero mean, this is done as:

$$\hat{x}_{k+1}^- = \phi_k \hat{x}_k \qquad (5.5.23).$$

- we are now in a position to repeat the whole procedure recursively.

- First, we will find the estimation error and error covariance matrix associated with the new "aged" prediction $\hat{x}_{k+1}^-$:

$$\begin{aligned} e_{k+1}^- &= x_{k+1} - \hat{x}_{k+1}^- \\ &= (\phi_k x_k + w_k) - \phi_k \hat{x}_k \\ &= \phi_k (x_k - \hat{x}_k) + w_k \\ &= \phi_k e_k + w_k \qquad (5.5.24) \end{aligned}$$

$$P_{k+1}^- = E\left[e_{k+1}^- e_{k+1}^{-T}\right]$$

$$= E\left[(\phi_k e_k + w_k)(\phi_k e_k + w_k)^T\right]$$

$$= E\left[(\phi_k e_k + w_k)(e_k^T \phi_k^T + w_k^T)\right]$$

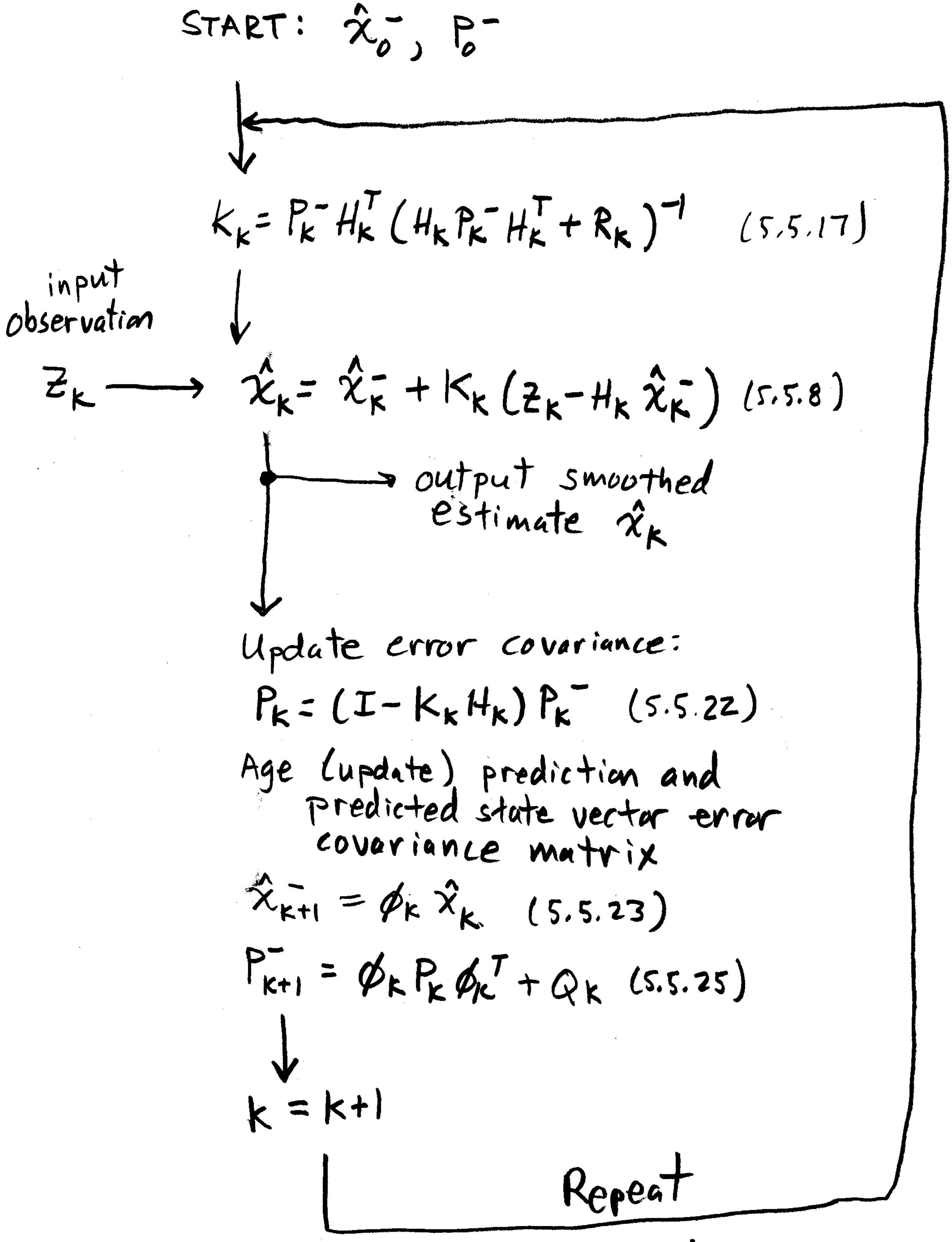
$$= E\left[\phi_k e_k e_k^T \phi_k^T + \phi_k e_k w_k^T + w_k e_k^T \phi_k^T + w_k w_k^T\right]$$

$$= \phi_k E\left[e_k e_k^T\right]\phi_k^T + \phi_k \underbrace{E\left[e_k w_k^T\right]}_{\text{zero}} + \underbrace{E\left[w_k e_k^T\right]}_{\text{zero}}\phi_k^T$$

$$+ E\left[w_k w_k^T\right]$$

$$= \phi_k P_k \phi_k^T + Q_k \qquad (5.5.25)$$

The recursive procedure:



START: $\hat{x}_0^-$, P_0^-

$$K_k = P_k^- H_k^T (H_k P_k^- H_k^T + R_k)^{-1} \quad (5.5.17)$$

input observation z_k

$$\hat{x}_k = \hat{x}_k^- + K_k (z_k - H_k \hat{x}_k^-) \quad (5.5.8)$$

output smoothed estimate $\hat{x}_k$

Update error covariance:

$$P_k = (I - K_k H_k) P_k^- \quad (5.5.22)$$

Age (update) prediction and predicted state vector error covariance matrix

$$\hat{x}_{k+1}^- = \phi_k \hat{x}_k \quad (5.5.23)$$

$$P_{k+1}^- = \phi_k P_k \phi_k^T + Q_k \quad (5.5.25)$$

$k = k+1$

Repeat

"Discrete Kalman Filter"

State Model:

Another View

$$x_{k+1} = \phi_k x_k + w_k$$

$$z_k = H_k x_k + v_k$$

x_k : $(n\times 1)$ state vector at time t_k

ϕ_k : $(n\times n)$ state transition matrix

w_k : $(n\times 1)$ forcing function

z_k : $(m\times 1)$ measurement vector

H_k : $(m\times m)$ output matrix

v_k : $(m\times 1)$ measurement noise

$E[w_k] = E[v_k] = 0$

$E[x_0 x_0^T] = \Pi_0$, given

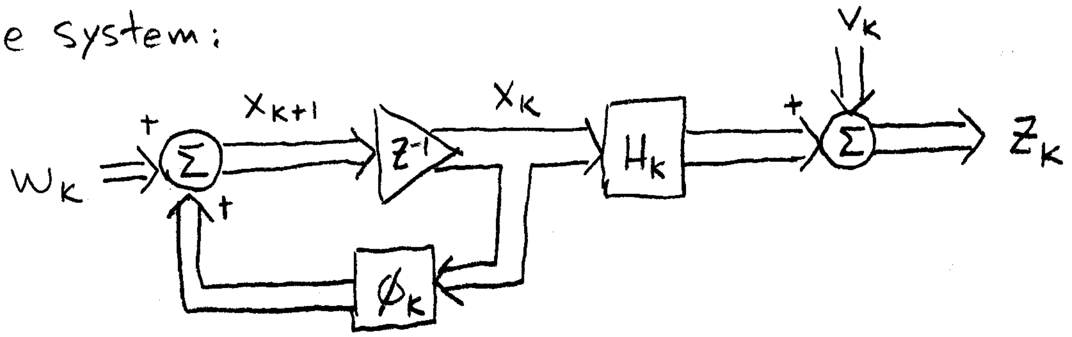
$E[x_0 v_k] = 0$

$E[w_k w_i^T] = Q_k \delta_{i-k}$

$E[v_k v_i^T] = R_k \delta_{i-k}$

$E[w_k v_i^T] = 0$

The system:



Kalman predictor update equation:

$$\hat{x}_{k+1}^- = \phi_k \hat{x}_k \qquad (5.5.23)$$

$$= \phi_k\left[\hat{x}_k^- + K_k(z_k - H_k \hat{x}_k^-)\right] \qquad (5.5.8)$$

$$= \phi_k K_k (z_k - H_k \hat{x}_k^-) + \phi_k \hat{x}_k^-$$

Predictor:

Let $\hat{x}_k^-$ = "a priori" estimate of x_k, given only k-1 observations

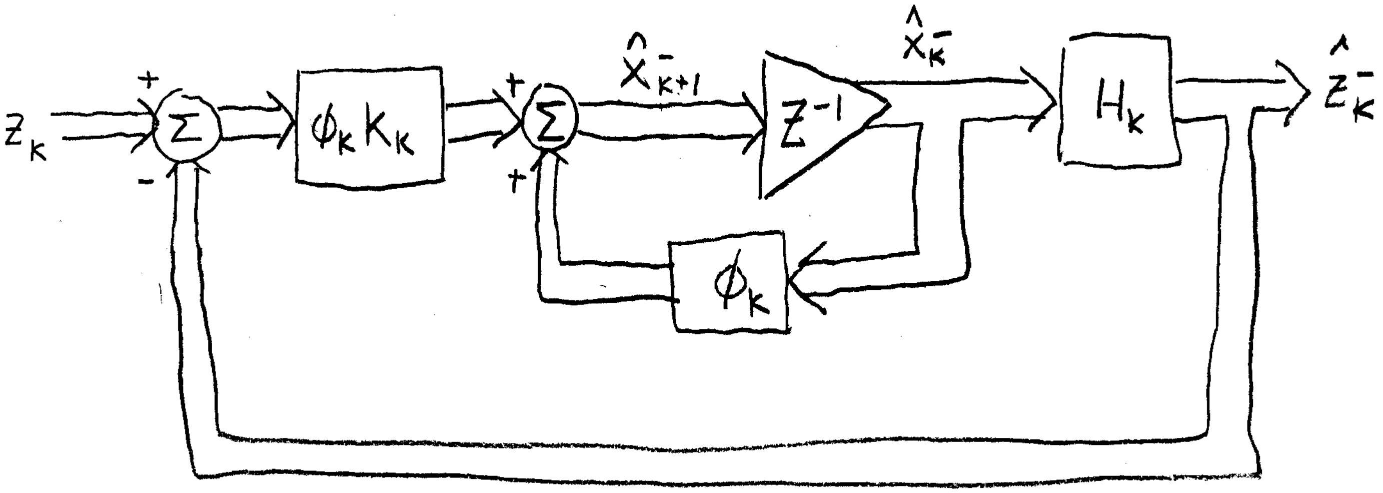
$= \hat{x}_{k|z_{k-1}, z_{k-2}, \ldots} = \hat{x}_{k|k-1}$.

Then $e_k^- = x_k - \hat{x}_{k-1}$: error in the predicted state vector.

- The predictor will turn out to be a system that looks much like the actual system.

→ It will have a state vector... namely $\hat{x}_k^-$.

The Predictor:



→ $\hat{x}_{k+1}^- = \phi_k K_k (z_k - H_k \hat{x}_k^-) + \phi_k \hat{x}_k^-$ (from page 5.33)

The filter: also has a state vector $\hat{x}_k$.

$$\hat{x}_{k+1} = \hat{x}^-_{k+1} + K_{k+1}\left(\overset{\substack{\text{non causal term}\\ \downarrow}}{z_{k+1}} - H_{k+1}\hat{x}^-_{k+1}\right) \qquad (5.5.8)$$

$$= \underset{\substack{\uparrow\\ (5.5.23)}}{\phi_k \hat{x}_k} + K_{k+1}\left[z_{k+1} - H_{k+1}\underset{\substack{\uparrow\\ (5.5.23)}}{\phi_k \hat{x}_k}\right]$$

$$= K_{k+1}\left(z_{k+1} - H_{k+1}\phi_k\hat{x}_k\right) + \phi_k\hat{x}_k$$

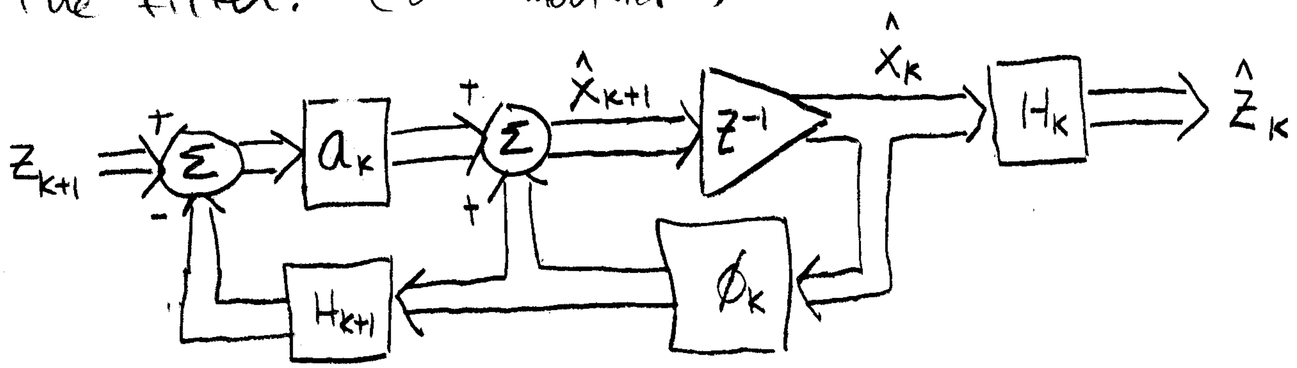
$$= \underbrace{P^-_{k+1}H^T_{k+1}\left(H_{k+1}P^-_{k+1}H^T_{k+1} + R_{k+1}\right)^{-1}}_{(5.5.17)}\left(z_{k+1} - H_{k+1}\phi_k\hat{x}_k\right) + \phi_k\hat{x}_k$$

$$= \left(\underset{\substack{\uparrow\\ (5.5.25)}}{\phi_k P_k \phi_k^T} + Q_k\right)H^T_{k+1}\left[H_{k+1}\left(\underset{\substack{\uparrow\\ (5.5.25)}}{\phi_k P_k \phi_k^T} + Q_k\right)H^T_{k+1} + R_{k+1}\right]^{-1}$$
$$\times\left(z_{k+1} - H_{k+1}\phi_k\hat{x}_k\right)$$
$$+ \phi_k\hat{x}_k$$

$$= \underbrace{\left(\phi_k P_k \phi_k^T H^T_{k+1} + Q_k H^T_{k+1}\right)\left(H_{k+1}\phi_k P_k \phi_k^T H^T_{k+1} + H_{k+1}Q_k H^T_{k+1} + R_{k+1}\right)^{-1}}_{\overset{\text{def}}{=}\ a_k}\left(z_{k+1} - H_{k+1}\phi_k\hat{x}_k\right)$$
$$+ \phi_k\hat{x}_k$$

$$= a_k\left(z_{k+1} - H_{k+1}\phi_k\hat{x}_k\right) + \phi_k\hat{x}_k$$

The filter: (or "smoother")



→ The noncausality will have to be addressed.

- we must solve for the optimal K_k and a_k.

K_k depends on P_k^-, H_k, and R_k.

→ H_k and R_k are <u>known</u>.

→ we must still solve for P_k^- ☆

a_k depends on ϕ_k, P_k, H_{k+1}, Q_k, R_{k+1}.

→ ϕ_k, H_{k+1}, Q_k, R_{k+1} are <u>known</u>.

→ we must still solve for P_k. ☆

→

Using the innovations representation and the orthogonality principle, it is possible to develop complete and independent representations for the filter and the predictor.

→ This involves expressing P_{k+1}^- as a function of P_k^- and expressing P_{k+1} as a function of P_k.

⇒ The book does not do this, so we will skip it.

- We can get around:

1. The "noncausility" problem with the filter.
2. The problem of not having independent recursive formulations for P_{k+1} in terms of P_k and P_{k+1}^- in terms of P_k^-.

→ By making use of the equations that relate quantities in the filter and the predictor to each other:

$$\hat{x}_k = \hat{x}_k^- + K_k(z_k - H_k \hat{x}_k^-) \qquad (5.5.8)$$

$$K_k = P_k^- H_k^T (H_k P_k^- H_k^T + R_k)^{-1} \qquad (5.5.17)$$

$$P_k = (I - K_k H_k) P_k^- \qquad (5.5.22)$$

$$P_{k+1}^- = \phi_k P_k \phi_k^T + Q_k \qquad (5.5.25)$$

$$\hat{x}_{k+1}^- = \phi_k \hat{x}_k \qquad (5.5.23)$$

Page 5.37

⇒ This is what is usually done in practical implementations. It leads to the procedure in Fig. 5.8 of the book.

Initialization

- In some cases, the dynamics of the system being estimated may give insight into the problem of initializing $\hat{x}_0^-$ and P_0^-. More on this later...

- In other cases, we may be able to deduce or may be given the initial state vector covariance matrix

$$\Pi_0 = E\left[x_0 x_0^T\right].$$

→ Then we can take

$$\hat{x}_0^- = 0$$
$$P_0^- = \Pi_0 .$$

- An alternative procedure can be used <u>if</u> we consider the first observation to be z_{-1} <u>and</u> <u>if</u> Π_{-1} is known <u>and</u> <u>if</u> z_{-1} and x_{-1} are known to be zero-mean Gaussians.

- In this case, it may be shown with some effort that

$$E(x_{-1}|z_{-1}) = E(x_{-1}) + \mathrm{Cov}(x_{-1}, z_{-1})\,\mathrm{Cov}^{-1}(z_{-1})\left[z_{-1} - E(z_{-1})\right]$$
$$= \mathrm{Cov}(x_{-1}, z_{-1})\,\mathrm{cov}^{-1}(z_{-1})\,z_{-1}$$

⟶

- we then take

$$
\begin{aligned}
\hat{X}_{-1} &= E(X_{-1} | Z_{-1}) \\
&= E(X_{-1} Z_{-1}^T)\{E(z^{-1} z^{-1T})\}^{-1} Z_{-1} \\
&= E[X_{-1}(H_{-1}X_{-1} + V_{-1})^T]\{E[(H_{-1}X_{-1} + V_{-1})(H_{-1}X_{-1} + V_{-1})^T]\}^{-1} Z_{-1} \\
&= E[X_{-1}(X_{-1}^T H_{-1}^T + V_{-1}^T)]\{E[(H_{-1}X_{-1} + V_{-1})(X_{-1}^T H_{-1}^T + V_{-1}^T)]\}^{-1} Z_{-1} \\
&= E[X_{-1}X_{-1}^T H_{-1}^T + X_{-1}V_{-1}^T]\{E[H_{-1}X_{-1}X_{-1}^T H_{-1}^T + H_{-1}X_{-1}V_{-1}^T \\
&\qquad + V_{-1}X_{-1}^T H_{-1}^T + V_{-1}V_{-1}^T]\}^{-1} Z_{-1} \\
&= (\pi_{-1}H_{-1}^T + 0)\{H_{-1}\pi_{-1}H_{-1}^T + 0 + 0 + R_{-1}\}^{-1} Z_{-1} \\
&= \underline{\pi_{-1}H_{-1}^T[H_{-1}\pi_{-1}H_{-1}^T + R_{-1}]^{-1} Z_{-1}}.
\end{aligned}
$$

- It may also be shown using innovations and orthogonality that

$$
\begin{aligned}
P_{-1} &= \pi_{-1} - E[\hat{X}_{-1}\hat{X}_{-1}^T] \\
&= \pi_{-1} - \pi_{-1}H_{-1}^T[H_{-1}\pi_{-1}H_{-1}^T + R_{-1}]^{-1}H_{-1}\pi_{-1}
\end{aligned}
$$

- $\hat{X}_0^-$ and P_0^- are then obtained from $\hat{X}_{-1}$ and P_{-1} using

$$\hat{X}_{K+1}^- = \phi_K \hat{X}_K \qquad (5.5.23)$$

$$P_{K+1}^- = \phi_K P_K \phi_K^T + Q_K \qquad (5.5.25)$$

Example

- we have an infrared camera mounted in a ball gimbal on an aircraft.
- It is a military aircraft, and we expect the enemy to fire threatening infrared tracking missiles at the aircraft.
- Our objective is to detect and track the missiles in our images.
- In each image where a missile is present, we want to compute two things:
  1) Our best estimate of the missile position in frame k given k observations... so we have the best possible battlefield situational awareness.
  2) Our best estimate of where the missile position will be in frame k+1 given k observations, so we can point the ball gimbal to maintain target lock.

⇒ Obviously, this problem is perfect for Kalman filtering.

Assumptions:

1. We will acquire the target when it is still far away. Thus, the target will appear as a single very bright pixel, perhaps surrounded by several relatively dimmer pixels that are somewhat brighter than the backgraund.

2. The frame rate of our camera is very fast compared to the kinematics of the target. Thus, the target is capable of only negligible accelerations from frame to frame.

3. The horizontal and vertical motions of the target are independent. Thus, we can consider the "x" and "y" observations of the target to be noisy measurements of the outputs of two independent systems.

Target Extraction:

- we will first apply a nonlinear filter to detect the target.
  - The main idea is this: at each pixel, formulate an estimate of the background. Subtract the background estimate from the pixel.

$I_{i-1,j-1}$	$I_{i,j-1}$	$I_{i-1,j+1}$
$I_{i-1,j}$	$I_{i,j}$	$I_{i+1,j}$
$I_{i-1,j+1}$	$I_{i,j+1}$	$I_{i+1,j+1}$

$$\begin{aligned}\text{Background Estimate} &= \text{MED}\left[\text{Eight Nearest Neighbors}\right] \\ &= \text{MED}\left[I_{i-1,j-1}, I_{i,j-1}, I_{i+1,j-1}, I_{i-1,j}, I_{i+1,j}, \right. \\ &\qquad \left. I_{i-1,j+1}, I_{i,j+1}, I_{i+1,j+1}\right] \\ &= \text{Average of the "middle" two eight-neighbors} \\ &= M_{i,j}.\end{aligned}$$

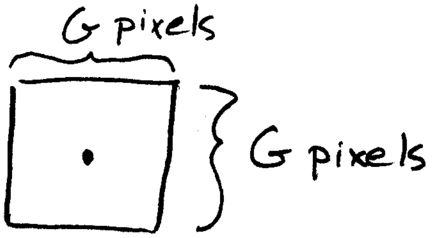
Filter output: $Y_{i,j} = I_{i,j} - M_{i,j}$.

- The median filtered image is then thresholded.
  - we could use an absolute threshold, a relative threshold (based, e.g., on the variance of the neighborhood), or both.
  - Pixels that pass the threshold are called "exceedances". They are candidate targets, denoted $E_{i,j}$.
- Each time we receive a new frame, there are two things that need to be done:
  1) Associate exceedances with existing tracks and use the associated exceedances as observations to update the existing tracks.
  2) Consider starting new tracks on any exceedances that do not associate with already existing tracks.

Gating:

- The "track gate" is a rectangular window centered on the estimated position of an existing track or a candidate new track.
- Let G be the width and height of the gate:

Gate:



- Typically, the gate size is varied during tracking.
- We will stipulate minimum and maximum values for G.
- Typically, for a new track, G starts out at G_{max}.
- why do this? (vary the gate size)
  - For a candidate new track, our knowledge of the true track position is typically poor. So we want a large, or "loose" gate to be sure we pick up all the exceedances associated with the track.
  - For an existing track, if the measurements have not been agreeing well with the predictions, we also want a large gate for the same reasons as above.
  - For an existing track where the measurements and predictions have been in good agreement, however, we want the gate to be small, or "tight".

⇒ Since only exceedances within the gate will be included in the observed threshold calculation, a tight gate means less noise in the centroid calculation.

Note: Henceforth, we consider only tracking in the horizontal direction. Vertical tracking is analogous.

Gating and Association: Update of existing tracks:

- For each existing track, we have a predicted position $\hat{i}^{c-}_{k} = \hat{i}^{c}_{k|k-1}$.
- When frame k arrives, we apply the spatial filter and threshold(s) to detect exceedances.
- For each track, we place the track gate at the predicted location $\hat{i}^{c-}_{k}$ and compute the centroid of the exceedances in the gate. This centroid is the observation z_k for the track.
- The measurement noise arises from the discrete nature of the camera focal plane array detector, thermal noise, and imperfections in the gating and target extraction processing.

Centroid Calculation:

$$z_K = \overset{\text{"observed"}}{\overset{\downarrow}{i^{c,o}_{K}}} = \frac{\displaystyle\sum_{m,n \in Gate} m\, E_{m,n}}{\displaystyle\sum_{m,n \in Gate} E_{m,n}} ,$$

$Y_{i,j}$ = spatial filter output.

$$\text{where} \quad E_{i,j} = \begin{cases} Y_{i,j} , & Y_{i,j} \geq \text{Threshold} \\ 0 , & \text{otherwise.} \end{cases}$$

Note: the standard deviation of the exceedances in the gate is also sometimes used in the calculation of G, the gate size.

- For each existing track, $i^{c,o}_{K}$ is used as the observation z_K to update the Kalman filter

New Track Starts:

- In any given frame, there may be exceedances that do not fall near the gate of any existing track (i.e., do not "associate".)
- A default sized gate is placed around such exceedances and they become candidates for new tracks.

5:45

- Since spurious bright spots may occur due to reflections, sun glints, etc., we don't start a new track right away.
- Instead, we require the candidate track to persist for some small number of frames before we actually start a new track. This is called the "persistance test" or the "temporal correspondence test".

Track Coasting:

- Sometimes a target is temporarily obscured.
- In this case, there will be no exceedances in the track gate of the existing track.
- When this occurs, we need to "coast" the track.
- This involves two steps:
  1. Open the track gate up to the maximum size.
  2. Allow the state equation to evolve with no input.
- If the number of missed detections exceeds a set limit, then the track is terminated. This is called "track loss" or "track deletion".
- It may be helpful to restart the Kalman filter upon reacquisition.

Track Crossings:

- Sometimes two targets will cross one another.
- When this happens, the track gates overlap.
- In this case, it may not be possible to resolve the individual targets.
- The best thing we can do is coast both tracks and hope to reacquire the targets when they separate.
- It may be helpful to restart the Kalman gains upon reacquisition.

Track Merge:

- Sometimes two tracks will cross and never come apart again.
- In this case, we merge the tracks.
- The simplest way to do this is to keep the track that best agrees with the observations and delete the other track.
- This is called a "track merge".
- It should be noted in the history of the deleted track that it was deleted due to a merge. Later offline analysis of both tracks may reveal insight into what actually happened.

Track Filter (Kalman Filter):

- The horizontal position of the target is i^c_k.
- We denote the horizontal velocity of the target in frame k by $\frac{d}{dt} i^c_k = v_k$.
- Since we are assuming a "constant velocity" model from frame to frame, the position (horizontal) of the target in frame k+1 is given by

$$i^c_{k+1} = i^c_k + \Delta v_k, \quad (*) \quad (\Delta = \text{frame time}).$$

- Now, it is obvious that, over many frames, the velocity cannot remain truly constant. Thus, we allow a small "velocity drift" from frame to frame:

$$v_{k+1} = v_k + u_k \, , \quad (**)$$

where u_k = white "velocity drift" noise with

$$E[u_k] = 0$$
$$E[u_k u_i] = \sigma_v^2 \delta_{k-i}$$

- Since we have used the symbol "v_k" for the target velocity, we will use "n_k" for the measurement noise (a white noise process), with

$$E[n_k] = 0 \qquad E[w_k n_i] = 0.$$
$$E[n_k n_i] = \sigma_n^2 \delta_{k-i}$$

- The observation is then given by $z_k = i^c_k + n_k$. $(***)$

- The system model is obtained by writing equations (*), (**), and (***) on page 5.48 together in matrix form:

$$\begin{bmatrix} i^c_{k+1} \\ v_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i^c_k \\ v_k \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k$$

$$z_k = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} i^c_k \\ v_k \end{bmatrix} + n_k \,.$$

- with reference to equations (5.5.1) and (5.5.2) in the book, we have

$$\phi_k = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix} \qquad x_k = \begin{bmatrix} i^c_k \\ v_k \end{bmatrix}$$

$$w_k = \begin{bmatrix} 0 \\ u_k \end{bmatrix}$$

$$H_k = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\underset{\substack{\uparrow \\ \text{“book”}}}{v_k} = n_k$$

- with reference to equations (5.5.3) and (5.5.4) in the book, we have

$$Q_k = E\left\{ \begin{bmatrix} 0 \\ u_k \end{bmatrix} \begin{bmatrix} 0 & u_i \end{bmatrix} \right\} = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_v^2 \delta_{i-k} \end{bmatrix}$$

$$R_k = E\left[n_k n_i \right] = \sigma_n^2 \delta_{k-i}$$

- At $k=-2$, we measure $z_{-2} = i^c_{-2} + n_{-2}$.
- At $k=-1$, we measure $z_{-1} = i^c_{-1} + n_{-1}$.

$\Rightarrow$ We take $\hat{i}^c_{-1} = z_{-1}$

and $\hat{v}_{-1} = \dfrac{z_{-1} - z_{-2}}{\Delta}$ ← note: the averaging reduces the effects of the measurement noise n_k.

- This gives us

$$\hat{x}_{-1} = \begin{bmatrix} \hat{i}^c_{-1} \\ \hat{v}_{-1} \end{bmatrix} = \begin{bmatrix} z_{-1} \\ \dfrac{z_{-1} - z_{-2}}{\Delta} \end{bmatrix}.$$

- Now,

note: NOT e^-_{-1} !!

$$P_{-1} = E\left[e_{-1} e_{-1}^T\right] = E\left\{ (x_{-1} - \hat{x}_{-1})(x_{-1} - \hat{x}_{-1})^T \right\}$$

$$= E\left\{ \left(\begin{bmatrix} i^c_{-1} \\ v_{-1} \end{bmatrix} - \begin{bmatrix} \hat{i}^c_{-1} \\ \hat{v}_{-1} \end{bmatrix} \right) \left(\begin{bmatrix} i^c_{-1} \\ v_{-1} \end{bmatrix} - \begin{bmatrix} \hat{i}^c_{-1} \\ \hat{v}_{-1} \end{bmatrix} \right)^T \right\}$$

$$= E\left\{ \begin{bmatrix} i^c_{-1} - \hat{i}^c_{-1} \\ v_{-1} - \hat{v}_{-1} \end{bmatrix} \begin{bmatrix} i^c_{-1} - \hat{i}^c_{-1} \\ v_{-1} - \hat{v}_{-1} \end{bmatrix}^T \right\}. \quad (*)$$

⟶

Now,

$$i^c_{-1} - \hat{i}^c_{-1} = i^c_{-1} - z_{-1}$$

$$= i^c_{-1} - (i^c_{-1} + n_{-1})$$

$$= -n_{-1} \quad (*)$$

$$V_{-1} - \hat{V}_{-1} = V_{-1} - \frac{z_{-1} - z_{-2}}{\Delta}$$

$$= v_{-1} - \frac{(i^c_{-1} + n_{-1}) - (i^c_{-2} + n_{-2})}{\Delta}$$

$$= V_{-1} - \frac{(i^c_{-1} - i^c_{-2}) + (n_{-1} - n_{-2})}{\Delta}$$

$$= v_{-1} - \frac{[(i^c_{-2} + \Delta v_{-2}) - i^c_{-2}] + (n_{-1} - n_{-2})}{\Delta}$$

$$= (V_{-2} + u_{-2}) - \frac{1}{\Delta}(\Delta v_{-2} + n_{-1} - n_{-2})$$

$$= V_{-2} + u_{-2} - V_{-2} - \frac{n_{-1} - n_{-2}}{\Delta}$$

$$= u_{-2} - \frac{n_{-1} - n_{-2}}{\Delta} \quad (**)$$

⟶

- We next use the results on page (19) to compute the quantities needed in equation (*) on page (18):

$$E\left[(i^c_{-1}-\hat{i}^c_{-1})^2\right] = E\left[(-n_{-1})^2\right] = E\left[n^2_{-1}\right] = \sigma_n^2 \quad (*)$$

$$E\left[(i^c_{-1}-\hat{i}^c_{-1})(v_{-1}-\hat{v}_{-1})\right] = E\left[(-n_{-1})\left(u_{-2}-\frac{n_{-1}-n_{-2}}{\Delta}\right)\right]$$

$$= E\left[-n_{-1}u_{-2} + \frac{n_{-1}}{\Delta}(n_{-1}-n_{-2})\right]$$

$$= E\left[-n_{-1}u_{-2} + \frac{n^2_{-1}}{\Delta} - \frac{n_{-1}n_{-2}}{\Delta}\right]$$

$$= -E\left[\cancelto{0}{n_{-1}u_{-2}}\right] + \frac{1}{\Delta}E\left[n^2_{-1}\right] - \frac{1}{\Delta}E\left[\cancelto{0}{n_{-1}n_{-2}}\right]$$

$$= \frac{\sigma_n^2}{\Delta} \quad (**)$$

$$E\left[(v_{-1}-\hat{v}_{-1})(i^c_{-1}-\hat{i}^c_{-1})\right] = E\left[(i^c_{-1}-\hat{i}^c_{-1})(v_{-1}-\hat{v}_{-1})\right] = \frac{\sigma_n^2}{\Delta} \quad (**)$$

→

$$E\left[(v_{-1}-\hat{v}_{-1})^2\right] = E\left[\left(u_{-2} - \frac{n_{-1}-n_{-2}}{\Delta}\right)^2\right]$$

$$= E\left[u_{-2}^2 - \frac{2u_{-2}}{\Delta}(n_{-1}-n_{-2}) + \frac{(n_{-1}-n_{-2})^2}{\Delta^2}\right]$$

$$= E\left[u_{-2}^2 - \frac{2}{\Delta}(u_{-2}n_{-1} - u_{-2}n_{-2}) + \frac{1}{\Delta^2}(n_{-1}^2 - 2n_{-1}n_{-2} + n_{-2}^2)\right]$$

$$= E\left[u_{-2}^2\right] - \frac{2}{\Delta}E\left[\cancelto{0}{u_{-2}n_{-1}}\right] + \frac{2}{\Delta}E\left[\cancelto{0}{u_{-2}n_{-2}}\right] + \frac{1}{\Delta^2}E\left[n_{-1}^2\right]$$

$$- \frac{2}{\Delta^2}E\left[\cancelto{0}{n_{-1}n_{-2}}\right] + \frac{1}{\Delta^2}E\left[n_{-2}^2\right]$$

$$= \sigma_v^2 + \frac{1}{\Delta^2}\sigma_n^2 + \frac{1}{\Delta^2}\sigma_n^2$$

$$= \frac{2\sigma_n^2}{\Delta^2} + \sigma_v^2 \quad (***)$$

So,

$$P_{-1} = \text{PAGE } 5.50\ (*)$$

$$= \begin{bmatrix} 5.52(*) & 5.52(**) \\ 5.52(**) & 5.53(***) \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_n^2 & \frac{\sigma_n^2}{\Delta} \\ \frac{\sigma_n^2}{\Delta} & \frac{2\sigma_n^2}{\Delta^2} + \sigma_v^2 \end{bmatrix}$$

- Now,

$$\hat{x}_0^- = \phi_{-1}\hat{x}_{-1} \qquad (5.5.23)$$

$$= \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{i}^c_{-1} \\ \hat{v}_{-1} \end{bmatrix}$$

$$P_0^- = \phi_{-1}P_{-1}\phi_{-1}^T + Q_{-1} \qquad (5.5.25)$$

$$= \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sigma_n^2 & \frac{\sigma_n^2}{\Delta} \\ \frac{\sigma_n^2}{\Delta} & \frac{2\sigma_n^2}{\Delta^2}+\sigma_v^2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \Delta & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \sigma_v^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2\sigma_n^2 & \frac{3\sigma_n^2}{\Delta}+\Delta\sigma_v^2 \\ \frac{\sigma_n^2}{\Delta} & \frac{2\sigma_n^2}{\Delta^2}+\sigma_v^2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \Delta & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \sigma_v^2 \end{bmatrix}$$

$$= \begin{bmatrix} 5\sigma_n^2+\Delta^2\sigma_v^2 & \frac{3\sigma_n^2}{\Delta}+\Delta\sigma_v^2 \\ \frac{3\sigma_n^2}{\Delta}+\Delta\sigma_v^2 & \frac{2\sigma_n^2}{\Delta^2}+\sigma_v^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \sigma_v^2 \end{bmatrix}$$

$$= \begin{bmatrix} 5\sigma_n^2+\Delta^2\sigma_v^2 & \frac{3\sigma_n^2}{\Delta}+\Delta\sigma_v^2 \\ \frac{3\sigma_n^2}{\Delta}+\Delta\sigma_v^2 & 2\left(\frac{\sigma_n^2}{\Delta^2}+\sigma_v^2\right) \end{bmatrix}.$$

⇒ We are now ready to enter the Kalman filtering loop given in Fig. 5.8 on page 219 of the book.

- We will now look more carefully at the equations in Fig. 5.8 and see that, in this case (constant velocity target model), they reduce to the well-known α-β filter:

- we begin with

$$\hat{x}_k = \hat{x}_k^- + K_k\left(z_k - H_k \hat{x}_k^-\right) \qquad (5.5.8)$$

$$\begin{bmatrix} \hat{i}^c_k \\ \hat{v}_k \end{bmatrix} = \begin{bmatrix} \hat{i}^{c-}_k \\ \hat{v}_k^- \end{bmatrix} + K_k\left(z_k - [1\ 0]\begin{bmatrix} \hat{i}^{c-}_k \\ \hat{v}_k^- \end{bmatrix}\right)$$

$$\begin{bmatrix} \hat{i}^c_k \\ \hat{v}_k \end{bmatrix} = \begin{bmatrix} \hat{i}^{c-}_k \\ \hat{v}_k^- \end{bmatrix} + K_k\left(\underbrace{z_k - \hat{i}^{c-}_k}_{\hookrightarrow \text{Disagreement between observed position and predicted position.}}\right)$$

- We now write $K_k = \begin{bmatrix} \alpha_k \\ \beta_k \end{bmatrix}$ to obtain <u>scalar</u> equations for the "smoothed" or "updated" estimates of the state vector entries:

$$\left\{ \begin{aligned} \hat{i}^c_k &= \hat{i}^{c-}_k + \alpha_k\left(z_k - \hat{i}^{c-}_k\right) \\ \hat{v}_k &= \hat{v}_k^- + \beta_k\left(z_k - \hat{i}^{c-}_k\right) \end{aligned} \right. \qquad (*)$$

→

- The "update" or "aging" equation for the predicted state vector is

$$\hat{x}_{k+1}^{-} = \phi_k \hat{x}_k \qquad (5.5.23)$$

- For our "constant velocity" model, this gives us

$$\begin{bmatrix} \hat{i}^{c-}_{k+1} \\ \hat{v}^{-}_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{i}^{c}_{k} \\ \hat{v}_{k} \end{bmatrix}$$

$$= \begin{bmatrix} \hat{i}^{c}_{k} + \Delta \hat{v}_k \\ \hat{v}_k \end{bmatrix}$$

- Or, as scalar equations,

$$\left\{ \begin{array}{l} \hat{i}^{c-}_{k+1} = \hat{i}^{c}_{k} + \Delta \hat{v}_k \\ \hat{v}^{-}_{k+1} = \hat{v}_k \end{array} \right. \qquad (*)$$

- Combining (*) above with (*) on P.5.55 we obtain the classical α-β filter equations:

$$\left\{ \begin{array}{l} \hat{i}^{c}_{k} = \hat{i}^{c-}_{k} + \alpha_k (z_k - \hat{i}^{c-}_{k}) \\ \hat{v}_k = \hat{v}^{-}_{k} + \beta_k (z_k - \hat{i}^{c-}_{k}) \\ \hat{i}^{c-}_{k+1} = \hat{i}^{c}_{k} + \Delta \hat{v}_k \\ \hat{v}^{-}_{k+1} = \hat{v}_k \end{array} \right. \qquad (**)$$

– Now we must address the computation of the so-called "Kalman gains" α_K and β_K.

NOTE:

→ P_0^- on page 5.54 does not depend on the observations z_K.

$$\rightarrow \left\{ \begin{aligned} K_K &= P_K^- H_K^T (H_K P_K^- H_K^T + R_K)^{-1} && (5.5.17) \\ P_K &= (I - K_K H_K) P_K^- && (5.5.22) \qquad (*) \\ P_{K+1}^- &= \Phi_K P_K \phi_K^T + Q_K && (5.5.25) \end{aligned} \right.$$

⇒ The Kalman gains depend on the predicted state vector error covariance matrix, but not directly on the observations.

⇒ The predicted state vector error covariance matrix depends on the filtered state vector error covariance matrix, but not directly on the observations.

⇒ The filtered state vector error covariance matrix depends on the predicted state vector error covariance matrix and on the Kalman gains, but not directly on the observations.

⇒ Thus, starting with the expression for P_0^- on page 5.54 and iterating equations (*) on page 5.57, we see that <u>none</u> of the Kalman gains and state vector error covariance matrices will <u>ever</u> depend on the observations.

⇒ This means that we can <u>PRECOMPUTE</u> all the Kalman gains offline and store them in a file.

⇒ Implementation of the Kalman filter is then reduced to processing the observations z_k using equations (**) on page 5.56 subject to the initial conditions

$$\left\{ \begin{array}{l} \hat{i}_{-1}^c = z_{-1} \\ \hat{v}_{-1} = \dfrac{z_{-1} - z_{-2}}{\Delta} \end{array} \Rightarrow \hat{x}_{-1} = \begin{bmatrix} z_{-1} \\ \dfrac{z_{-1}-z_{-2}}{\Delta} \end{bmatrix} \right.$$

$$\hat{x}_0^- = \phi_{-1} \hat{x}_{-1} = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{i}_{-1}^c \\ \hat{v}_{-1} \end{bmatrix}$$

given on pages 5.50 and 5.54.

Computation of the Gains:

- Let $P_k = \begin{bmatrix} P_k(1,1) & P_k(1,2) \\ P_k(2,1) & P_k(2,2) \end{bmatrix}$

$$P_k^- = \begin{bmatrix} P_k^-(1,1) & P_k^-(1,2) \\ P_k^-(2,1) & P_k^-(2,2) \end{bmatrix}$$

- The P_k^- iteration is initialized with the expression for P_0^- given on page 5.55

- Kalman gains:

$$K_k = P_k^- H_k^T (H_k P_k^- H_k^T + R_k)^{-1} \quad (5.5.17)$$

$$\begin{bmatrix} \alpha_k \\ \beta_k \end{bmatrix} = \begin{bmatrix} P_k^-(1,1) & P_k^-(1,2) \\ P_k^-(2,1) & P_k^-(2,2) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left([1 \;\; 0] \begin{bmatrix} P_k^-(1,1) & P_k^-(1,2) \\ P_k^-(2,1) & P_k^-(2,2) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sigma_n^2 \right)^{-1}$$

$$= \begin{bmatrix} P_k^-(1,1) \\ P_k^-(2,1) \end{bmatrix} \left([P_k^-(1,1) \;\; P_k^-(1,2)] \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sigma_n^2 \right)^{-1}$$

$$= \begin{bmatrix} P_k^-(1,1) \\ P_k^-(2,1) \end{bmatrix} \left(P_k^-(1,1) + \sigma_n^2 \right)^{-1}$$

$$= \begin{bmatrix} \dfrac{P_k^-(1,1)}{P_k^-(1,1) + \sigma_n^2} \\ \dfrac{P_k^-(2,1)}{P_k^-(1,1) + \sigma_n^2} \end{bmatrix} \longrightarrow$$

$$\left\{ \begin{aligned} \alpha_k &= \frac{P_k^-(1,1)}{P_k^-(1,1) + \sigma_n^2} \\ \beta_k &= \frac{P_k^-(2,1)}{P_k^-(1,1) + \sigma_n^2} \end{aligned} \right. \qquad (*)$$

- Filtered State Vector Error Covariance Matrix;

$$P_k = (I - K_k H_k) P_k^- \qquad (5.5.22)$$

$$\begin{bmatrix} P_k(1,1) & P_k(1,2) \\ P_k(2,1) & P_k(2,2) \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \alpha_k \\ \beta_k \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right) \begin{bmatrix} P_k^-(1,1) & P_k^-(1,2) \\ P_k^-(2,1) & P_k^-(2,2) \end{bmatrix}$$

$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \alpha_k & 0 \\ \beta_k & 0 \end{bmatrix} \right) \begin{bmatrix} P_k^-(1,1) & P_k^-(1,2) \\ P_k^-(2,1) & P_k^-(2,2) \end{bmatrix}$$

$$= \begin{bmatrix} 1-\alpha_k & 0 \\ -\beta_k & 1 \end{bmatrix} \begin{bmatrix} P_k^-(1,1) & P_k^-(1,2) \\ P_k^-(2,1) & P_k^-(2,2) \end{bmatrix}$$

$$= \begin{bmatrix} (1-\alpha_k) P_k^-(1,1) & (1-\alpha_k) P_k^-(1,2) \\ P_k^-(2,1) - \beta_k P_k^-(1,1) & P_k^-(2,2) - \beta_k P_k^-(1,2) \end{bmatrix} \qquad (**)$$

→

- Predicted State Vector Error Covariance Matrix:

$$P_{k+1}^- = \phi_k P_k \phi_k^T + Q_k \qquad (5.5.25)$$

$$\begin{bmatrix} P_{k+1}^-(1,1) & P_{k+1}^-(1,2) \\ P_{k+1}^-(2,1) & P_{k+1}^-(2,2) \end{bmatrix} = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_k(1,1) & P_k(1,2) \\ P_k(2,1) & P_k(2,2) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \Delta & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \sigma_v^2 \end{bmatrix}$$

$$= \begin{bmatrix} P_k(1,1) + \Delta P_k(2,1) & P_k(1,2) + \Delta P_k(2,2) \\ P_k(2,1) & P_k(2,2) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \Delta & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \sigma_v^2 \end{bmatrix}$$

$$\left\{ \begin{aligned} P_{k+1}^-(1,1) &= P_k(1,1) + \Delta P_k(2,1) + \Delta P_k(1,2) + \Delta^2 P_k(2,2) = P_k(1,1) + 2\Delta P_k(1,2) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (*) \quad + \Delta^2 P_k(2,2) \\ P_{k+1}^-(1,2) &= P_k(1,2) + \Delta P_k(2,2) \\ P_{k+1}^-(2,1) &= P_k(2,1) + \Delta P_k(2,2) = P_k(1,2) + \Delta P_k(2,2) \\ P_{k+1}^-(2,2) &= P_k(2,2) + \sigma_v^2 \end{aligned} \right.$$

<u>Note</u>: For covariance Matrices, the (2,1) element and (1,2) element must be equal.

⟶

⇒ Thus, to precompute the gains:

```
1. Begin with $P_0^-$ as given on page 5.54
2. $k=0$
    3. Compute $\alpha_k$ and $\beta_k$ using (*) on page 5.60
    4. Compute $P_k$ using (**) on page 5.60
    5. Compute $P_{k+1}^-$ using (*) on page 5.61
    6. $k=k+1$
```

- Typically, the Kalman gains will rapidly converge to asymptotic values.

- The iteration above need only be carried out until convergence occurs.

Asymptotic Behavior

- we would like to know for sure when the Kalman gains have converged.
- Thus, it is of interest to compute the assymptotic values of the gains.

- Eq. (*) on 5.60 can be substituted into (**) on 5.60 to obtain P_k in terms of P_k^-.

- The above result can be substituted into (*) on 5.61 to get P_{k+1}^- in terms of P_k^-.

- In the limit as $k \to \infty$, $P_{k+1}^- = P_k^- = P_\infty^-$. Plugging this into the above gives three equations in three unknowns: $P_\infty^-(1,1)$, $P_\infty^-(2,2)$, and $P_\infty^-(1,2) = P_\infty^-(2,1)$.
- These can be solved. The solution can then be plugged into (*) on 5.60 to solve for the asymptotic gains, α_∞ and β_∞.

Nonwhite Forcing Function

- This example is from Section 5.7 of the book.

- Shipboard INS; we wish to track INS drift.

ψ_x = longitude position error (E-w)

ψ_y = latitude position error (N-S)

ψ_z = [azimuthal error] $- \psi_x \tan(\text{latitude})$

$\Omega_x = \Omega \cos(\text{latitude})$

$\Omega_z = \Omega \sin(\text{latitude})$

x = North axis
y = West axis
z = up axis.

$\varepsilon_x, \varepsilon_y, \varepsilon_z$ = INS gyro drift rates.

→

- The error dynamics are given by

$$\begin{aligned}\dot{\psi}_x &= \Omega_z \psi_y + \varepsilon_x \\ \dot{\psi}_y &= -\Omega_z \psi_x + \Omega_x \psi_z + \varepsilon_y \\ \dot{\psi}_z &= -\Omega_x \psi_y + \varepsilon_z\end{aligned}$$

- For state variables, we take ψ_x, ψ_y, ψ_z.

- we could write the state model

$$\begin{bmatrix} \dot{\psi}_x \\ \dot{\psi}_y \\ \dot{\psi}_z \end{bmatrix} = \begin{bmatrix} 0 & \Omega_z & 0 \\ -\Omega_z & 0 & \Omega_x \\ 0 & -\Omega_x & 0 \end{bmatrix} \begin{bmatrix} \psi_x \\ \psi_y \\ \psi_z \end{bmatrix} + \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{bmatrix}.$$

- The problem with this is: the drift rates $\varepsilon_x, \varepsilon_y, \varepsilon_z$ change very slowly in time.

→ They are each highly self-correlated over short time intervals.

→ So the above model doesn't satisfy the "uncorrelated forcing function" requirement of the Kalman filter.

- But, $\dot{\varepsilon}_x$, $\dot{\varepsilon}_y$, $\dot{\varepsilon}_z$ <u>are</u> expected to be uncorrelated.

$$\dot{\varepsilon}_x = f_x$$
$$\dot{\varepsilon}_y = f_y$$
$$\dot{\varepsilon}_z = f_z$$

- We get around the problem by adding ε_x, ε_y, ε_z to the state vector. This is called "Augmenting the state vector".

$$\begin{bmatrix} \dot{\psi}_x \\ \dot{\psi}_y \\ \dot{\psi}_z \\ \dot{\varepsilon}_x \\ \dot{\varepsilon}_y \\ \dot{\varepsilon}_z \end{bmatrix} = \begin{bmatrix} 0 & \Omega_z & 0 & 1 & 0 & 0 \\ -\Omega_z & 0 & \Omega_x & 0 & 1 & 0 \\ 0 & -\Omega_x & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_x \\ \psi_y \\ \psi_z \\ \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ f_x \\ f_y \\ f_z \end{bmatrix}$$

- Generally, for a nonwhite forcing function, we can find the "shaping filter" that relates the nonwhite function to white noise.

- We can then use white noise as the system input, incorporate the nonwhite forcing function into the state vector, and add the shaping filter equations to the system equations.

- A similar procedure can be used for nonwhite measurement noise.

  1. Remove nonwhite v_k entries from

$$z_k = H_k x_k + v_k$$

  2. Put the removed entries into the state vector.

  3. Add white noise entries to w_k and add the required shaping filter equations to the system equations.

CONDITIONAL DENSITY VIEWPOINT

- Section 5.8 of the book starts to hint around the actual original Kalman filter solution a little bit.

- While it is not said in the book, this is again related to the innovations process and the orthogonality principle.

⇒ Chap. 5 up to here in a nutshell:

1. Usual Kalman filter problem statement
2. Assume the a priori prediction $\hat{x}_0^-$ is given.
3. Assume that $\hat{x}_k$ is <u>linearly</u> related to $\hat{x}_k^-$ and z_k. Find MMSE gains.

⇒ This is a quite restricted notion of the Kalman filter. In fact, the Kalman filter is much more powerful and much more profound than the above implies

- Here is the development given in the book with some added clarifications:

- Assume the standard Kalman problem statement, and assume further that the input noise w_k and measurement noise v_k are <u>Gaussian</u>.

- Let z_k^* be the vector of all observations from time 0 up until time k:

$$z_k^* = [z_0 \; z_1 \; \cdots \; z_k]^T$$

- Keeping all the notation unchanged from earlier in the chapter, we have the true state vector x_k and the filtered estimate $\hat{x}_k$ (as yet unspecified).

- NOTE: $\hat{x}_k$ doesn't depend on the true x_k. $\hat{x}_k$ depends only on the observations z_k^*.
  → So, conditioned on z_k^*, $\hat{x}_k$ is deterministic, not stochastic.

- The MSE of $\hat{x}_k$ conditioned on z_k^* is

$$E\left[(x_k - \hat{x}_k)^T (x_k - \hat{x}_k) \,\middle|\, z_k^*\right] = E\left[x_k^T x_k - x_k^T \hat{x}_k + \hat{x}_k^T x_k + \hat{x}_k^T \hat{x}_k \,\middle|\, z_k^*\right]$$

→

$$= E\left[X_K^T X_K | Z_K^*\right] - E\left[X_K^T | Z_K^*\right]\hat{X}_K - \hat{X}_K^T E\left[X_K | Z_K^*\right] + \hat{X}_K^T \hat{X}_K \quad (5.8.1)$$

(complete the square)

$$= E\left[X_K^T X_K | Z_K^*\right] + \left\{\hat{X}_K - E\left[X_K | Z_K^*\right]\right\}^T \left\{\hat{X}_K - E\left[X_K | Z_K^*\right]\right\} - E\left[X_K^T | Z_K^*\right] E\left[X_K | Z_K^*\right] \quad (5.8.2)$$

$\Rightarrow$ only the middle term depends on $\hat{x}_K$.

$\Rightarrow$ In designing an optimal $\hat{x}_K$, we can only hope to impact the middle term.

$\Rightarrow$ The middle term is minimized by choosing

$$\hat{x}_K = E\left[X_K | Z_K^*\right], \quad (5.8.3)$$

the conditional mean of x_K given Z_K^*.

- Let $x_{k|k-1} = x_k$ "given" z^*_{k-1}

$= x_k$ conditioned on z^*_{k-1}.

- Let $P^-_{k|k-1} = E\left[(x_{k|k-1} - \hat{x}^-_k)(x_{k|k-1} - \hat{x}^-_k)^T\right]$.

- Then $x_{k|k-1} \underset{\uparrow}{\sim} \mathcal{N}(\hat{x}^-_k, P^-_{k|k-1})$ (5.8.4)

"is distributed"

- Since $z_k = H_k x_k + v_k$ (5.8.5) = (5.5.2)

we have $z_k \sim \mathcal{N}(H_k \hat{x}^-_k, H_k P^-_{k|k-1} H_k^T + R_k)$

- Also, it is clear that

$$P(z_k | x_{k|k-1}) \sim \mathcal{N}(H_k x_{k|k-1}, R_k) \quad (5.8.7)$$

→ b/c once $x_{k|k-1}$ is <u>known</u>, the error covariance matrix P^-_k contains all zeros.

⟶

- Applying Bayes rule,

$$p\left(x_{k|k-1} \middle| z_k\right) = \frac{p\left(z_k \middle| x_{k|k-1}\right) p\left(x_{k|k-1}\right)}{p\left(z_k\right)}$$

(5.8.8).

- Analytical expressions for each quantity on the RHS of (5.8.8) were given on PAGE 5.70.

⇒ Now, it may be seen that the conditional RV $x_{k|k-1} | z_k$ in (5.8.8) is identical to the optimal estimate

$$\hat{x}_k = E\left[x_k \middle| z_k^*\right] \quad (5.8.3)$$

on page 5.69.

→ The density /distribution of the optimal estimate $\hat{x}_k$ may be compute using (5.5.8) above.

There results:

$$E[\hat{x}_K] = \hat{x}_K^- + P_K^- H_K^T (H_K P_K^- H_K^T + R_K)^{-1} (z_K - H_K \hat{x}_K^-)$$

(5.8.10) = (5.5.8)+(5.5.17)

$$\mathrm{Cov}(\hat{x}_K) = \left[(P_K^-)^{-1} + H_K^T R_K^{-1} H_K \right]^{-1} \qquad (5.8.11)$$

$$\overset{\text{nontrivial}}{=} (I - K_K H_K) P_K^- \qquad (5.5.22).$$

⇒ This is the same answer we got before in Section 5.5.

- But here, we did not assume that the estimator had to be linear.

⇒ For Gaussian statistics on w_K and v_K, the Kalman filter is the optimal estimator period.

→ There is no better estimator, including all possible linear and nonlinear estimators.

→ Moreover, for Gaussian statistics,

$$\hat{x}_K = E\left[x_K \mid Z_K^*\right]$$

is also the Maximum Liklihood estimator.

- It occurs at the mode (maximum) of the distribution (which is unimodal].

- Therefore, $\hat{x}_K$ also minimizes the expected value of practically any nondecreasing function of $(x_K - \hat{x}_K)$.

⇒ Thus, under Gaussian statistics, the Kalman filter is optimal

- In the MMSE sense
- In the MMAE sense

⋮

etc.