Semi-tensor Product of Matrices and Its Applications

Daizhan Cheng

Institute of Systems Science
Academy of Mathematics and Systems Science
Chinese Academy of Sciences

January 14, 2011

Outline of Presentation

I. Semi-tensor Product of Matrices

☞ $A_{m\times n} \times B_{p\times q} = ?$

Definition 1.1

Let $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{p\times q}$. Denote

$$t := \mathrm{lcm}(n, p).$$

Then we define the semi-tensor product (STP) of A and B as

$$A \ltimes B := \left(A \otimes I_{t/n}\right)\left(B \otimes I_{t/p}\right) \in \mathcal{M}_{(mt/n)\times(qt/p)}. \tag{1}$$

☞ Some Basic Comments

- When $n = p$, $A \ltimes B = AB$. So the STP is a generalization of conventional matrix product.
- When $n = rp$, denote it by $A \succ_r B$;
  when $rn = p$, denote it by $A \prec_r B$.
  These two cases are called the multi-dimensional case, which is particularly important in applications.
- STP keeps almost all the major properties of the conventional matrix product unchanged.

☞ Examples

Example 1.2

1. Let $X = \begin{bmatrix} 1 & 2 & 3 & -1 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then

$$X \ltimes Y = \begin{bmatrix} 1 & 2 \end{bmatrix} \cdot 1 + \begin{bmatrix} 3 & -1 \end{bmatrix} \cdot 2 = \begin{bmatrix} 7 & 0 \end{bmatrix}.$$

2. Let $X = \begin{bmatrix} -1 & 2 & 1 & -1 & 2 & 3 \end{bmatrix}^T$ and $Y = \begin{bmatrix} 1 & 2 & -2 \end{bmatrix}$. Then

$$X \ltimes Y = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot 1 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot 2 + \begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot (-2) = \begin{bmatrix} -3 \\ -6 \end{bmatrix}.$$

Example 1.2 (Continued)

3. Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 1 & 2 \\ 3 & 2 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}.$$

Then

$$A \ltimes B = \begin{bmatrix} \begin{pmatrix} 1 & 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ -1 \end{pmatrix} \\ \begin{pmatrix} 2 & 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} & \begin{pmatrix} 2 & 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} -2 \\ -1 \end{pmatrix} \\ \begin{pmatrix} 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} & \begin{pmatrix} 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ -1 \end{pmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 & -3 & -5 \\ 4 & 7 & -5 & -8 \\ 5 & 2 & -7 & -4 \end{bmatrix}.$$

☞ Insight Meaning

Let $A \in \mathcal{M}_{m\times n}$. Consider a bilinear form

$$P(x,y) = x^T A y. \tag{2}$$

Set (Row Stacking Form)

$$V_r(A) = (a_{11}, \cdots, a_{1n}, \cdots, a_{m1}, \cdots, a_{mn}).$$

Then

$$P(x,y) = V_r(A) \ltimes x \ltimes y. \tag{3}$$

$\ltimes$ can search pointer mechanically!

☞ Multilinear Mapping

$$P : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}.$$

(? Cubic Matrix)

$$P(\delta_m^i, \delta_n^j, \delta_s^k) := r_{i,j,k},$$
$$i = 1, \cdots, m; j = 1, \cdots, n; k = 1, \cdots, s.$$

Define

$$M_P = [r_{111}, \cdots, r_{1,1,s} \cdots r_{mn1}, \cdots, r_{mns}].$$

Then

$$P(x, y, z) = M_P \ltimes x \ltimes y \ltimes z. \tag{4}$$

It is available for general multilinear mappings.

☞ Properties

Proposition 1.3

- (Distributive rule)

$$\begin{aligned} A \ltimes (\alpha B + \beta C) &= \alpha A \ltimes B + \beta A \ltimes C; \\ (\alpha B + \beta C) \ltimes A &= \alpha B \ltimes A + \beta C \ltimes A, \quad \alpha, \beta \in \mathbb{R}. \end{aligned} \tag{5}$$

- (Associative rule)

$$A \ltimes (B \ltimes C) = (A \ltimes B) \ltimes C. \tag{6}$$

Proposition 1.4

- $$(A \ltimes B)^T = B^T \ltimes A^T. \quad (7)$$

- Assume both A and B are invertible. Then

$$(A \ltimes B)^{-1} = B^{-1} \ltimes A^{-1}. \quad (8)$$

Proposition 1.5 (Pseudo-Commutativity)

Assume $A \in \mathcal{M}_{m\times n}$ is given.

- Let $Z \in \mathbb{R}^t$ be a row vector. Then

$$A \ltimes Z = Z \ltimes (I_t \otimes A); \tag{9}$$

- Let $Z \in \mathbb{R}^t$ be a column vector. Then

$$Z \ltimes A = (I_t \otimes A) \ltimes Z. \tag{10}$$

☞ Remarks

- Let $\xi \in \mathbb{R}^n$ be a column (row). Then

$$\xi^k := \underbrace{\xi \ltimes \cdots \ltimes \xi}_{k}.$$

- Let $A \in \mathcal{M}_{m\times n}$ and $m|n$ or $n|m$. Then

$$A^k := \underbrace{A \ltimes \cdots \ltimes A}_{k}.$$

- In Boolean algebra, all matrices $A \in \mathcal{M}_{m\times n}$, where $m = 2^p$ and $n = 2^q$ (or for k-valued case: $m = k^p$ and $n = k^q$), which is the multiple dimensional case.

☞ Swap Matrix

Definition 1.6

A swap matrix, $W_{[m,n]}$ is an $mn \times mn$ matrix constructed in the following way: label its columns by $(11, 12, \cdots, 1n, \cdots, m1, m2, \cdots, mn)$ and its rows by $(11, 21, \cdots, m1, \cdots, 1n, 2n, \cdots, mn)$. Then its element in the position $((I,J),(i,j))$ is assigned as

$$w_{(IJ),(ij)} = \delta_{i,j}^{I,J} = \begin{cases} 1, & I = i \text{ and } J = j, \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

When $m = n$ we briefly denote $W_{[n]} := W_{[n,n]}$.

☞ Example

Example 1.7

Let $m = 2$ and $n = 3$, the swap matrix $W_{[2,3]}$ is constructed as

$$W_{[2,3]} = \begin{array}{c} \begin{matrix} (11) & (12) & (13) & (21) & (22) & (23) \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{matrix} (11) \\ (21) \\ (12) \\ (22) \\ (13) \\ (23) \end{matrix} .$$

☞ Properties

Proposition 1.8

- Let $X \in \mathbb{R}^m$ and $Y \in \mathbb{R}^n$ be two columns. Then

$$W_{[m,n]} \ltimes X \ltimes Y = Y \ltimes X, \quad W_{[n,m]} \ltimes Y \ltimes X = X \ltimes Y. \tag{12}$$

- Let $A \in \mathcal{M}_{m\times n}$. Then

$$W_{[m,n]} V_r(A) = V_c(A), \quad W_{[n,m]} V_c(A) = V_r(A). \tag{13}$$

- Let $X_i \in \mathbb{R}^{n_i}$, $i = 1, \cdots, m$. Then

$$\begin{aligned} &\left(I_{n_1+\cdots+n_{k-1}} \otimes W_{[n_k,n_{k+1}]} \otimes I_{n_{k+2}+\cdots+n_m}\right) \\ &X_1 \ltimes \cdots \ltimes X_k \ltimes X_{k+1} \ltimes \cdots \ltimes X_m \\ &= X_1 \ltimes \cdots \ltimes X_{k+1} \ltimes X_k \ltimes \cdots \ltimes X_m. \end{aligned} \tag{14}$$

☞ Properties

Proposition 1.9

- The swap matrix is an orthogonal matrix as
$$W_{[m,n]}^T = W_{[m,n]}^{-1} = W_{[n,m]}. \tag{15}$$
-
$$W_{[m,n]} = \left(\delta_n^1 \ltimes \delta_m^1 \quad \cdots \quad \delta_n^n \ltimes \delta_m^1 \quad \cdots\cdots \quad \delta_n^n \ltimes \delta_m^m\right), \tag{16}$$
where δ_n^i is the ith column of I_n.

☞ "$\times$" VS "$\ltimes$"

	CP $\times$	STP $\ltimes$
Property	Similar	Similar
Applicability	linear, bilinear	multilinear
Commutativity	No	Pseudo-Commutative

☞ My Book

II. Boolean Network

Kaffman: for cellular networks, gene regulatory networks, etc.

☞ Network Graph

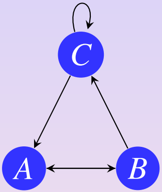


Figure 1: A Boolean network

☞ Network Dynamics

$$\begin{cases} A(t+1) = B(t) \wedge C(t) \\ B(t+1) = \neg A(t) \\ C(t+1) = B(t) \vee C(t) \end{cases} \quad (17)$$

Boolean Control Network

☞ Network Graph

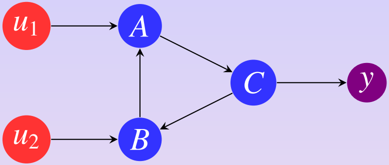


Figure 2: A Boolean control network

☞ Network Dynamics

Its logical equation is

$$\begin{cases} A(t+1) = B(t) \wedge u_1(t) \\ B(t+1) = C(t) \vee u_2(t) \\ C(t+1) = A(t) \end{cases} \tag{18}$$
$$y(t) = \neg C(t)$$

☞ Dynamics of Boolean Network

$$\begin{cases} x_1(t+1) = f_1(x_1(t), \cdots, x_n(t)) \\ \vdots \\ x_n(t+1) = f_n(x_1(t), \cdots, x_n(t)), \quad x_i \in \mathcal{D}, \end{cases} \tag{19}$$

where

$$\mathcal{D} := \{0, 1\}.$$

☞ Dynamics of Boolean Control Network

$$\begin{cases} x_1(t+1) = f_1(x_1(t), \cdots, x_n(t), u_1(t), \cdots, u_m(t)) \\ \vdots \\ x_n(t+1) = f_n(x_1(t), \cdots, x_n(t), u_1(t), \cdots, u_m(t)), \end{cases}$$
$$y_j(t) = h_j(x(t)), \quad j = 1, \cdots, p, \tag{20}$$

where $x_i, u_i, y_i \in \mathcal{D}$.

☞ Some Notations

- $\mathcal{D} = \{0 \sim \text{False}, 1 \sim \text{True}\}$;
- $\mathbf{1}_k := (\underbrace{1\ 1\ \cdots\ 1}_{k})^T$;
- δ_n^i: the i-th column of I_n;
- $\Delta_n := \{\delta_n^i | i = 1, \cdots, n\}$, $\Delta := D_2$;
- A matrix $L \in \mathcal{M}_{n \times r}$ is called a logical matrix if

$$\mathrm{Col}(L) \subset \Delta_n.$$

  Denote by $\mathcal{L}_{n \times r}$ the set of $n \times r$ logical matrices.
- Let $L = [\delta_n^{i_1}, \delta_n^{i_2}, \cdots, \delta_n^{i_r}] \in \mathcal{L}_{n \times r}$. Briefly,

$$L = \delta_n[i_1, i_2, \cdots, i_r].$$

☞ Vector Form of Logical Mapping

$$1 \sim \delta_2^1,\ 0 \sim \delta_2^2 \quad \Rightarrow \quad \mathcal{D} \sim \Delta.$$

- Logical function:

$$f : \mathcal{D}^n \to \mathcal{D} \quad \Rightarrow \quad \Delta^n \to \Delta;$$

- Logical mapping:

$$F : \mathcal{D}^n \to \mathcal{D}^m \quad \Rightarrow \quad \Delta^n \to \Delta^m.$$

The later function (mapping) is called the vector form.

☞ Structure Matrix (1)

Theorem 2.1

Let $y = f(x_1, \cdots, x_n) : \Delta^n \to \Delta$. Then there exists unique $M_f \in \mathcal{L}_{2\times 2^n}$ such that

$$y = M_f x, \quad \text{where } x = \ltimes_{i=1}^n x_i. \tag{21}$$

Definition 2.2

The M_f is called the **structure matrix** of f.

☞ Structure Matrix (2)

Theorem 2.3

Let $F : \Delta^n \to \Delta^k$ be defined by

$$y_i = f_i(x_1, \cdots, x_n).$$

Then there exists unique $M_F \in \mathcal{L}_{2^k \times 2^n}$ such that

$$y = M_F x, \tag{22}$$

where

$$x = \ltimes_{i=1}^{n} x_i; \qquad y = \ltimes_{i=1}^{k} y_i.$$

Definition 2.4

The M_F is called the **structure matrix** of F.

☞ Structure Matrices of Logical Operators

Table 1: Structure Matrices of Logical Operators

$\neg$	M_n	$\delta_2[2\ 1]$
$\vee$	M_d	$\delta_2[1\ 1\ 1\ 2]$
$\wedge$	M_c	$\delta_2[1\ 2\ 2\ 2]$
$\rightarrow$	M_i	$\delta_2[1\ 2\ 1\ 1]$
$\leftrightarrow$	M_e	$\delta_2[1\ 2\ 2\ 1]$
$\bar{\vee}$	M_p	$\delta_2[2\ 1\ 1\ 2]$

☞ Matrix Expression of Subspace

- State Space: $\mathcal{X} = F_\ell(x_1, \cdots, x_n)$
- Subspace: $\mathcal{V} = F_\ell(y_1, \cdots, y_k)$, $y_i \in \mathcal{X}$ is described by

$$y_i = f_i(x_1, \cdots, x_n), \quad i = 1, \cdots, k.$$

- Algebraic Form:

$$y = F_v x,$$

  where

$$x = \ltimes_{i=1}^n x_i,\ y = \ltimes_{i=1}^k y_i,\ F_v \in \mathcal{L}_{2^k \times 2^n}.$$

- Conclusion: Each $F_v \in \mathcal{L}_{2^k \times 2^n}$ uniquely determines a subspace $\mathcal{V}$.

☞ Algebraic Form of BN (19)

$$x(t+1) = Lx(t), \tag{23}$$

where $L \in \mathcal{L}_{2^n \times 2^n}$.

☞ Algebraic Form of BCN (20)

$$\begin{cases} x(t+1) = Lu(t)x(t) \\ y(t) = Hx(t), \end{cases} \tag{24}$$

where $L \in \mathcal{L}_{2^n \times 2^{n+m}}$, $H \in \mathcal{L}_{2^p \times 2^n}$.

Algebraic Form

Example

Example 2.5

- Consider Boolean network (17) in Fig. 1. We have

$$L = \delta_8[3\ 7\ 7\ 8\ 1\ 5\ 5\ 6].$$

- Consider Boolean control network (18) in Fig. 2. We have

$$\begin{aligned} L &= \delta_8[1\ 1\ 5\ 5\ 2\ 2\ 6\ 6\ 1\ 3\ 5\ 7\ 2\ 4\ 6\ 8 \\ &\qquad 5\ 5\ 5\ 5\ 6\ 6\ 6\ 6\ 5\ 7\ 5\ 7\ 6\ 8\ 6\ 8]; \\ H &= \delta_2[2\ 1\ 2\ 1\ 2\ 1\ 2\ 1]. \end{aligned}$$

☞ Topological Structure

- Find “fixed points”, “cycles”;
- Find “basin of attraction” ,“transient time”;
- “Rolling Gear” structure, which explains why “tiny attractors” decide “vast order”.

References:

D. Cheng, H. Qi, A linear representation of dynamics of Boolean networks, *IEEE Trans. Aut. Contr.*, vol. 55, no. 10, pp. 2251-2258, 2010. (Regular Paper)

D. Cheng, Input-state approach to Boolean networks, *IEEE Trans. Neural Networks*, vol. 20, no. 3, pp. 512-521, 2009. (Regular Paper)

☞ Basic Control Properties

- Controllability under open-loop or closed-loop controls;
- Observability;
- Algebraic description of input-output transfer graph.

References:

D. Cheng, H. Qi, Controllability and observability of Boolean control networks, *Automatica*, vol. 45, no. 7, pp. 1659-1665, 2009. (Regular Paper)

Y. Zhao, H. Qi, D. Cheng, Input-state incidence matrix of Boolean control networks and its applications, *Sys. Contr. Lett.*, vol. 46, no. 12, pp. 767-774, 2010.

☞ System Realization

- State space expression;
- Input-output realization;
- Kalman decomposition, minimum realization.

References:

D. Cheng, Z. Li, H. Qi, Realization of Boolean control networks, *Automatica*, vol. 46, no. 1, pp. 62-69, 2010. (Regular Paper)

D. Cheng, H. Qi, State space analysis of Boolean network, *IEEE Trans. Neural Networks*, vol. 21, no. 4, pp. 584-594, 2010. (Regular Paper)

☞ Control Design

- Disturbance decoupling;
- Stability and stabilization;
- Canalizing mapping and its applications.

References:

D. Cheng, Disturbance Decoupling of Boolean control networks, *IEEE Trans. Aut. Contr.*, 2011. (to appear) (Regular Paper)

D. Cheng, H. Qi, Z. Li, J.B. Liu, Stability and stabilization of Boolean networks, *Int. J. Robust Nonlin. Contr.*, doi:10.1002/rnc.1581 (to appear).

☞ Optimal Control

- Topological structure of Boolean control networks;
- Optimal control and its design.
- k- and Mix-valued and higher-order control networks.

References:

Y. Zhao, Z. Li, D. Cheng, Optimal control of logical control networks *IEEE Trans. Aut. Contr.*, (to appear) (Regular Paper).

Z. Li, D. Cheng, Algebraic approach to dynamics of multi-valued networks, *Int. J. Bifurcat. Chaos*, vol. 20, no. 3, pp. 561-582, 2010.

☞ Identification

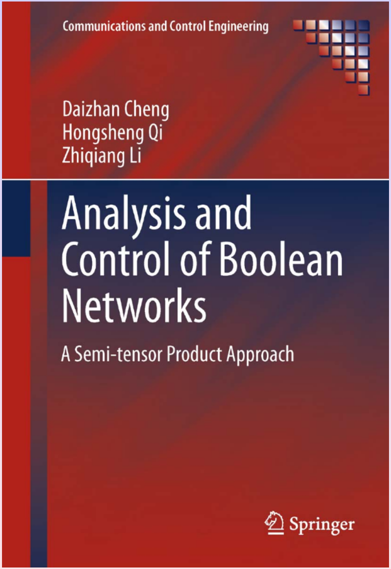
- Identify the dynamic evolution;
- Identify via input-output data.

References:

D. Cheng, Y. Zhao, Identification of Boolean control networks, *Automatica*, (to appear) (Regular Paper).

D. Cheng, H. Qi, Z. Li, Model construction of Boolean network via observed data, *IEEE Trans. Neural Networks*, (to appear) (Regular Paper).

☞ My Book


Communications and Control Engineering
Daizhan Cheng
Hongsheng Qi
Zhiqiang Li
Analysis and Control of Boolean Networks
A Semi-tensor Product Approach
Springer

III. Continuous Dynamic Systems

☞ Differential

Definition 3.1

Let $A(x) = (a_{ij}(x)) \in \mathcal{M}_{p\times q}$ be a matrix with entries as smooth functions of $x \in \mathbb{R}^n$. Its differential $DA(x) \in \mathcal{M}_{p\times nq}$ is constructed by replacing $a_{ij}(x)$ by its differential

$$da_{ij}(x) = \begin{bmatrix} \frac{\partial a_{ij}(x)}{\partial x_1} & \frac{\partial a_{ij}(x)}{\partial x_2} & \cdots & \frac{\partial a_{ij}(x)}{\partial x_n} \end{bmatrix}.$$

☞ Properties

Proposition 3.2 (Product Rule)

$$D\left[A(x)B(x)\right] = DA(x) \ltimes B(x) + A(x) \ltimes DB(x). \tag{25}$$

Proposition 3.3 (Basic Formula)

Define

$$\Phi_k = \sum_{s=0}^{k} I_{n^s} W_{[n^{k-s},n]}.$$

Then

$$D(x^{k+1}) = \Phi_k x^k. \tag{26}$$

☞ Taylor Expansion

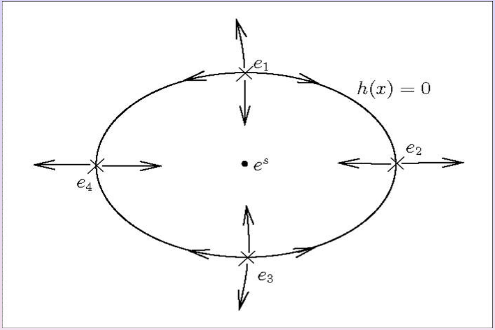
Theorem 3.4 (Taylor Expansion)

Let $f(x) = f(x_1, \cdots, x_n)$ be a smooth function. Then

$$f(x) = f(0) + D(f)(0)x + \frac{1}{2!}D^2f(0)x^2 + \cdots . \qquad (27)$$

☞ Stability Region

$$\dot{x} = f(x) = F_1 x + F_2 x^2 + F_3 x^3 + \cdots, \quad x \in \mathbb{R}^n.$$



Stability boundary is composed of the stable sub-manifolds of the unstable equilibriums on the boundary.

☞ Formula for Stability Region

Theorem 3.5

Let the boundary be $h(x) = 0$. Then $h(x)$ is uniquely determined by

$$\begin{cases} h(0) = 0 \\ h(x) = \eta^T x = O(\|x\|^2) \\ L_f h(x) = \mu h(x), \end{cases} \tag{28}$$

where
η: eigenvector w.r.t. positive eigenvalue of $J_f(0)$.
μ: non-zero parament.

☞ Calculation of Lie Derivative

$$
\begin{aligned}
L_f h &= Dh \cdot f \\
&= D\left(H_0 + H_1 x + H_2 x^2 + \cdots\right) \cdot f \\
&= \left(H_1 + H_2 \Phi_1 x + H_3 \Phi_2 x^2 + \cdots\right)\left(F_1 x + F_2 x^2 + \cdots\right) \\
&= H_1 F_1 x + H_1 F_2 x^2 + H_2 \Phi_1 x F_1 x + \cdots \\
&= H_1 F_1 x + \left[H_1 F_2 + H_2 \Phi_1 (I_n \otimes F_1)\right] x^2 + \cdots \\
&:= C_1 x + C_2 x^2 + C_3 x^3 + \cdots
\end{aligned}
$$

☞ Main Result

Theorem 3.6

$$h(x) = H_1 x + \frac{1}{2} x^t \Psi x + H_3 x^3 + \cdots , \tag{29}$$

where

$$\begin{cases} H_1 = \eta^T \\ \Psi = V_c^{-1} \left\{ \left[\left(\frac{\mu}{2} I_n - J^T \right) \otimes I_n + I_n \otimes \left(\frac{\mu}{2} I_n - J^T \right) \right]^{-1} \right. \\ \qquad \left. V_c \left(\sum_{i=1}^{n} \eta_i \mathrm{Hess}(f_i)(0) \right) \right\} \\ H_k = G_k T_B(n,k), \quad k \geq 3 \end{cases}$$

with

$$G_k = \left[\sum_{i=1}^{k-1} G_i T_B(n,i) \Phi_{i-1} (I_{n^{i-1}} \otimes F_{k-i+1}) \right] T_n(n,k) C_k^{-1}$$

$$C_k = \mu I_d - T_B(n,k) \Phi_{k-1} \left(I_{n^{k-1}} \otimes F_1 \right) T_N(n,k).$$

☞ Control Design

- Morgan's problem;
- Non-regular feedback linearization;
- Symmetry of nonlinear systems.

References:

- D. Cheng, Semi-tensor product of matrices and its application to Morgan's Problem, *Science in China, Series F*, vol. 44, no. 3, pp. 195-212, 2001.
- D. Cheng, X. Hu, Y. Wang, Non-regular feedback linerization of nonlinear systems, *Automatica*, vol. 40, no. 3, pp. 439-447, 2004.
- D. Cheng, J. Ma, Q. Lu, S. Mei, Quadratic form of stable sub-manifold for power systems, *Int. J. Rob. Nonlin. Contr.*, vol. 14, pp. 773-788, 2004.
- D. Cheng, G. Yang, Z. Xi, Nonlinear systems possessing linear symmetry, *Int. J. Rob. Nonlin. Contr.*, vol. 17, no. 1, pp. 51-81, 2007.

☞ Application to Power Systems

IV. Application to Math and Physics

☞ Lei Algebra

Definition 4.1

Let V be a real vector space with $*: V \times V \to V$.

- $(V, *)$ is called an algebra, if (distributivity)

$$(aX + bY) * Z = aX * Z + bY * Z, \quad a, b \in \mathbb{R},\ X, Y, Z \in V;$$

- An algebra is called a Lie algebra, if
  - **(i)** (skew symmetry)

$$X * Y = -Y * X;$$

  - **(ii)** (Jacobi Identity)

$$X * (Y * Z) + Y * (Z * X) + Z * (X * Y) = 0.$$

☞ Structure Matrix of a Algebra

Let $(V,*)$ be an algebra, and $\{e_1,\cdots,e_n\}$ a basis of V. Denote

$$e_i * e_j = c_{ij}^1 e_1 + c_{ij}^1 e_2 + \cdots + c_{ij}^n e_n, \quad i,j = 1,\cdots,n.$$

We construct a matrix, called the structure matrix of the algebra, as

$$M = \begin{bmatrix} c_{11}^1 & c_{12}^1 & \cdots & c_{1n}^1 & \cdots & c_{nn}^1 \\ c_{11}^2 & c_{12}^2 & \cdots & c_{1n}^2 & \cdots & c_{nn}^2 \\ \vdots & & & & & \\ c_{11}^n & c_{12}^n & \cdots & c_{1n}^n & \cdots & c_{nn}^n \end{bmatrix}.$$

☞ Product Using Structure Matrix

Proposition 4.2

Let M be the structure matrix of $(V, *)$. Then

$$X * Y = MXY. \tag{30}$$

Example 4.3

- $$(X * Y * Z) = (MXY) * Z = M(MXY)Z = M^2XYZ.$$
- $$\underbrace{X * X * \cdots * X}_{k} = M^{k-1}X^k.$$

☞ Verifying Lie Algebra

Theorem 4.4

Let L be an algebra with structure matrix $M_L \in \mathcal{M}_{n\times n^2}$. Then V is a Lie algebra, **iff**,

(i)

$$M_L\left(W_{[n]} + I_n\right) = 0;$$

(ii)

$$M_L^2\left(I_{n^2} + W_{[n^2,n]} + W_{[n,n^2]}\right) = 0.$$

☞ Cross Product on $\mathbb{R}^3$

Proposition 4.5

$(\mathbb{R}^3, \times)$ is a Lie algebra, where $\times$ is the cross product.

Its structure matrix is

$$M_\times = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Invoking Theorem 4.4, the proof is a straightforward computation.

☞ Any other Lie algebra(s) on $\mathbb{R}^3$?

Theorem 4.6

A three-dimensional algebra is a Lie algebra, iff, its structure matrix is as

$$M = \begin{bmatrix} 0 & a & d & -a & 0 & g & -d & -g & 0 \\ 0 & b & e & -b & 0 & h & -e & -h & 0 \\ 0 & c & f & -c & 0 & i & -f & -i & 0 \end{bmatrix},$$

with entries satisfying

$$\begin{cases} bg + gf - ah - di = 0 \\ ae - bd + hf - ei = 0 \\ af + bi - cd - ch = 0. \end{cases}$$

☞ Another Lie algebra on $\mathbb{R}^3$

Example 4.7

$(\mathbb{R}^3, *)$ is a Lie algebra, where

$$\begin{cases} \vec{i} * \vec{i} = \vec{j} * \vec{j} = \vec{k} * \vec{k} = 0 \\ \vec{i} * \vec{j} = -\vec{j} * \vec{i} = -7\vec{i} + 10\vec{j} - 11\vec{k} \\ \vec{i} * \vec{k} = -\vec{k} * \vec{i} = \vec{i} - \vec{j} + 2\vec{k} \\ \vec{j} * \vec{k} = -\vec{k} * \vec{j} = -2\vec{i} + 3\vec{j} - 3\vec{k}. \end{cases}$$

$$M_* = \begin{bmatrix} 0 & -7 & 1 & 7 & 0 & -2 & -1 & 2 & 0 \\ 0 & 10 & -1 & -10 & 0 & 3 & 1 & -3 & 0 \\ 0 & -11 & 2 & 11 & 0 & -3 & -2 & 3 & 0 \end{bmatrix}.$$

☞ Applications to Math and Physics

- Contraction of tensor field;
- Calculation of connection in Differential Geometry;
- Structure of algebras and fields.

References:

D. Cheng, Y. Dong, Semi-tensor product of matrices and its some applications to physics, *Meth. Appl. Analysis*, vol. 10, no. 4, pp. 565-588, 2003.

D. Cheng, Some applications of semi-tensor product of matrices in algebra, *Comp. Math. Appl.*, vol. 52, pp. 1045-1066, 2006.

V. Concluding Remarks

☞ Current Research Topics

- Game Theory:
  - Finite history strategy in dynamic game;
  - Evolutionary games on networks.
- Universal algebra:
  - Structure of lattice;
  - Structure matrix $\Rightarrow$ Homomorphism.
- Cryptography:
  - Symmetry of Boolean functions.
- Fuzzy control:
  - Solving fuzzy relational equations;
  - Design of multi-input fuzzy controllers.

☞ Remarks

- Semi-tensor product is a simple and useful tool;
- Numerical tool in computer era;
- It is with 100% originality;
- It has attracted international attention;
- You are expected to join us.

Please try it!

Thank you!

Question?