LYAPUNOV-KRASOVSKII STABILITY THEOREM FOR FRACTIONAL SYSTEMS WITH DELAY

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Fractional calculus techniques and methods started to be applied during the last decades in several fields of science and engineering. In this paper we studied the stability of fractional order nonlinear time-delay systems for Caputo's derivative and we extended Lyapunov-Krasovskii theorem for the fractional nonlinear systems.

Key words: fractional nonlinear systems, stability, Lyapunov-Krasovskii theorem, time-delay systems.

1. INTRODUCTION

Fractional calculus is an emerging field with various applications in science and engineering. Fractional calculus is a good candidate to solve the dynamics of complex systems. During the last years fractional calculus was subjected to an intense debate [1-4]. Fractional differential equations started to play an important role in modeling anomalous diffusion, processes having long range dependence and so on. Several open problems remain unsolved or there were partially solved with this type of calculus. Among those kinds of problems we mention the question of stability which is of main interest in control theory. Also, the problem of time-delay system has been discussed over many years. Time delay is very often encountered in different technical systems, *e.g.* electric, pneumatic and hydraulic networks, chemical processes, and long transmission lines. The existence of pure time delay, regardless of its presence in a control and/or state, may cause undesirable system transient response, or generally, even an instability. Numerous reports have been published on this matter, with particular emphasis on the application of Lyapunov's second method [5, 6].

In recent years, considerable attention has been paid to control systems whose processes and/or controllers are of fractional order. This is mainly due to the fact that many real-world physical systems are well characterized by fractional-order differential equations, *i.e.,* equations involving noninteger-order derivatives. In particular, it has been shown that viscoelastic materials having memory and hereditary effects [7] and dynamical processes such as semi-infinite lossy RC transmission [8], mass diffusion and heat conduction [9], can be more adequately modeled by fractional-order models than integer-order models. Moreover, with the success in the synthesis of real noninteger differentiator and the emergence of new electrical circuit element called "fractance" [10], fractional-order controllers [11, 12] including fractional-order PID controllers [13] have been proposed to enhance the robustness and performance of control systems.

Some literatures published about stability of fractional order linear time delay systems [14, 15]. In the base of Lyapunov's second method, some work has been done in the field of stability of fractional order nonlinear systems without delay [16-18]. But it seems that a few attentions have been paid to the stability of fractional order nonlinear time-delay systems.

The purpose of this paper is to develop the Lyapunov-Krasovskii theorem for fractional order nonlinear time-delay systems.

The manuscript is organized as follows: In Section 2 some basic definitions of fractional calculus are mentioned. Section 3 is devoted to fractional nonlinear time-delay systems. Section 4 presents the generalization of the fractional Lyapunov-Krasovskii theorem when both fractional derivatives and delay are presented. Finally, the Conclusions are shown in Section 5.

2. PRELIMINARIES AND DEFINITIONS

In the fractional calculus the Riemann-Liouville and Caputo fractional derivatives are defined respectively [15, 16]

$$ {}_{t_0}D_t^q x(t) = \frac{1}{\Gamma(n-q)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left(\int_{t_0}^{t} \frac{x(s)}{(t-s)^{q+1-n}} \mathrm{d}s \right), \qquad (n-1 < q \le n), \tag{1} $$

$$ {}_{t_0}^{c}D_t^q x(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} \frac{x^{(n)}(s)}{(t-s)^{q+1-n}} \mathrm{d}s, \qquad (n-1 < q \le n), \tag{2} $$

where $x(t)$ is an arbitrary differentiable function, $n \in \mathbb{N}$ and ${}_{t_0}D_t^q$ and ${}_{t_0}^{c}D_t^q$ are the Riemann-Liouville and Caputo fractional derivatives of order q on $[t_0, t_1]$ respectively, and $\Gamma(\cdot)$ denotes the Gamma function.

For $0 < q \le 1$ we have

$$ {}_{t_0}D_t^q x(t) = \frac{1}{\Gamma(1-q)} \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_{t_0}^{t} \frac{x(s)}{(t-s)^q} \mathrm{d}s \right), \qquad (0 < q \le 1) \tag{3} $$

and

$$ {}_{t_0}^{c}D_t^q x(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} \frac{x'(s)}{(t-s)^q} \mathrm{d}s, \qquad (0 < q \le 1). \tag{4} $$

Some properties of Riemann-Liouville and Caputo derivatives are recalled below [15, 16]:

Property 1.

When $0 < q < 1$, we have

$$ {}_{t_0}^{c}D_t^q x(t) = {}_{t_0}D_t^q x(t) - \frac{x(t_0)}{\Gamma(1-q)} (t-t_0)^{-q}. \tag{5} $$

In particular, if $x(t_0) = 0$, we have

$$ {}_{t_0}^{c}D_t^q x(t) = {}_{t_0}D_t^q x(t). \tag{6} $$

Property 2.

For and $v > -1$, we have

$$ {}_{t_0}D_t^q (t-t_0)^v = \frac{\Gamma(1+v)}{\Gamma(1+v-q)} (t-t_0)^{v-q}. \tag{7} $$

In particular, if $0 < q < 1$ and $x(t) = (t-t_0)^v$ then from Property 1, we have

$$ {}_{t_0}^{c}D_t^q (t-t_0)^v = \frac{\Gamma(1+v)}{\Gamma(1+v-q)} (t-t_0)^{v-q}. \tag{8} $$

Property 3.

$$ {}_{t_0}^{c}D_t^q \big(ax(t) + by(t)\big) = a\, {}_{t_0}^{c}D_t^q x(t) + b\, {}_{t_0}^{c}D_t^q y(t), \tag{9} $$

where a and b are arbitrary constants.

Property 4.
From the definition of Caputo's derivative (4) when $0 < q \le 1$ we have

$$I_{t_0}^{q}\, {}_{t_0}^{c}D_t^q x(t) = x(t) - x(t_0), \tag{10}$$

where $(I_{t_0}^{q} f)(t) = \dfrac{1}{\Gamma(q)} \displaystyle\int_{t_0}^{t} \dfrac{f(s)\mathrm{d}s}{(t-s)^{1-q}}\ , t > t_0, \mathfrak{A}(q) > 0.$

3. FRACTIONAL NONLINEAR TIME-DELAY SYSTEM

Let $C([a,b],\mathbb{R}^n)$ be the set of continuous functions mapping the interval $[a,b]$ to $\mathbb{R}^n$. In many situations, one may wish to identify a *maximum time delay* r of a system. In this case, we are often interested in the set of continuous function mapping $[-r,0]$ to $\mathbb{R}^n$, for which we simplify the notation to $C = C([-r,0],\mathbb{R}^n)$. For any $A > 0$ and any continuous function of time $\psi \in C([t_0 - r, t_0 + A],\mathbb{R}^n)$, $t_0 \le t \le t_0 + A$, let $\psi_t \in C$ be a segment of function ψ defined as $\psi_t(\theta) = \psi_t(t+\theta)$, $-r \le \theta \le 0$.

Consider Caputo fractional nonlinear time-delay system

$${}_{t_0}^{c}D_t^q x(t) = f(t, x_t), \tag{11}$$

where $x(t) \in \mathbb{R}^n, 0 < q < 1$ and $f : \mathbb{R} \times C \to \mathbb{R}^n$. As such, to determine the future evolution of the state. It is necessary to specify the initial state variables $x(t)$ in a time interval of length r, say, from $t_0 - r$ to t_0, *i.e.*,

$$x_{t_0} = \phi, \tag{12}$$

where $\phi \in C$ is given. In other words $x(t_0 + \theta) = \phi(\theta), -r \le \theta \le 0$.
for a function $\phi \in C([a,b],\mathbb{R}^n)$ define the continuous norm $\|\cdot\|_c$ by

$$\|\phi\|_c = \max_{a \le \theta \le b} \|\phi(\theta)\|. \tag{13}$$

Definition. For the system described by (11) the trivial solution $x(0) = 0$ is said to be stable if for any $t_0 \in \mathbb{R}$ and any $\varepsilon > 0$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $\|x_{t_0}\|_c < \delta$ implies $\|x(t)\| < \varepsilon$ for $t \ge t_0$. It is said to be asymptotically stable if it is stable and for any $t_0 \in \mathbb{R}$ and any $\varepsilon > 0$, there exists a $\delta_0 = \delta_0(t_0, \varepsilon) > 0$ such that $\|x_{t_0}\|_c < \delta_0$ implies $\lim_{x\to\infty} x(t) = 0$. It is said to be uniformly stable if it is stable and $\delta = \delta(\varepsilon) > 0$ can be chosen independently of t_0. It is uniformly asymptotically stable if it is uniformly stable and there exists a $\delta_0 > 0$ and functions $\delta(\varepsilon)$, $T(\varepsilon)$ such that $\|x_{t_0}\|_c < \delta_0$ and $t \geq t_0 + T(\varepsilon)$ implies $\|x(t)\| < \varepsilon$. It is globally (uniformly) asymptotically stable if it is (uniformly) asymptotically stable and δ_0 can be an arbitrary large, finite number [21].

4. FRACTIONAL LYAPUNOV-KRASOVSKII THEOREM

As in the study of systems without delay, an effective method for determining the stability of a time-delay system is Lyapunov method. Since in a time-delay system the "state" at time t required the value of $x(t)$ in the interval $[t-r, t]$, *i.e.*, x_t, it is natural to expect that for a time-delay system, corresponding Lyapunov function be a functional $V(t, x_t)$ depending on x_t, which also should measure the deviation of x_t from the trivial solution 0.

Let $V(t, \phi)$ be differentiable, and let $x_t(\tau, \phi)$ be the solution of (11) at time t with initial condition $x_\tau = \phi$. Then we calculate the Caputo derivative of $V(t, x_t)$ with respect to t and evaluate it at $t = \tau$ as follows

$$
\begin{aligned}
{}_{t_0}^{c}D_t^q V(\tau, \phi) &= {}_{t_0}^{c}D_t^q V(t, x_t(\tau, \phi))\Big|_{t=\tau, x_t=\phi} \\
&= \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} \frac{V'(s, x_s)}{(t-s)^q} \mathrm{d}s \Bigg|_{t=\tau, x_t=\phi},
\end{aligned}
\tag{14}
$$

where $0 < q < 1$.

Theorem: Suppose $f : \mathbb{R} \times C \to \mathbb{R}^n$ in (6) maps $\mathbb{R} \times (\textit{bounded sets in}\, C)$ into bounded sets in $\mathbb{R}^n$, and $\alpha_1, \alpha_2, \alpha_3 : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ are continuous nondecreasing functions, where additionally $\alpha_1(s), \alpha_2(s)$ are positive for $s > 0$, and $\alpha_1(0) = \alpha_2(0) = 0$. If there exists a continuously differentiable functional $V : \mathbb{R} \times S_\rho \to \mathbb{R}$, where $S_\rho = \{\phi \in C : \|\phi\|_C < \rho\}$, such that

$$
\alpha_1(\|\phi(0)\|) \leq V(t, \phi) \leq \alpha_2(\|\phi\|_c) \tag{15}
$$

and

$$ {}_{t_0}^{c}D_t^q V(t,\phi) \le -\alpha_3\left(\|\phi(0)\|\right), \quad 0<q\le 1. \tag{16}$$

Then the trivial solution of (11) is uniformly stable. If $\alpha_3(s)>0$ for $s>0$ then it is uniformly asymptotically stable. If, in addition, $\lim_{s\to\infty}\alpha_1(s)=\infty$, then it is globally uniformly asymptotically stable.

The integer order derivative version of this theorem can be found in [21, 22].

Proof. For any $\varepsilon>0$, since α_2 is continuous and $\alpha_2(0)=0$ we can find a sufficiently small $\delta=\delta(\varepsilon)>0$ such that $\alpha_2(\delta)<\alpha_1(\varepsilon)$. Hence, for any initial time t_0 and any initial condition $x_{t_0}=\phi$ with $\|\phi\|_c<\delta$, we have ${}_{t_0}^{c}D_t^q V(t,x_t)\le 0$ and therefore by property 4 $V(t,x_t)\le V(t_0,\phi)$, for any $t\ge t_0$. This implies that

$$\alpha_1\left(\|x(t)\|\right)\le V(t,x_t)\le V(t_0,\phi)\le \alpha_2\left(\|\phi\|_c\right)\le\alpha_2(\delta)<\alpha_1(\varepsilon), \tag{17}$$

which implies that $\|x(t)\|<\varepsilon$ for $t\ge t_0$. This proves the uniform stability.

To prove uniform asymptotic stability, let $0<\varepsilon<\rho$ and $\delta=\delta(\varepsilon)>0$ correspond to uniform stability. Choose an $\varepsilon_0\le\rho$ and designate by $\delta_0=\delta(\varepsilon_0)>0$ where ε_0 is fixed. Let us now choose $\|x_{t_0}\|_c\le\delta_0$ and $T(\varepsilon)=\left[\frac{\alpha_2(\delta_0)}{\alpha_3(\delta(\varepsilon))}\Gamma(1+q)\right]^{\frac{1}{q}}$ where $\delta(\varepsilon)$ corresponds to uniform stability. Suppose that $\|x_{t_0}\|_c\le\delta_0$ and we would have $\|x(t)\|\ge\delta(\varepsilon)$ for all $t\ge t_0$

$$-\alpha_3\left(\|x(t)\|\right)\le-\alpha_3\left(\delta(\varepsilon)\right). \tag{18}$$

Therefore,

$$ {}_{t_0}^{c}D_t^q V(t,x_t)\le-\alpha_3\left(\delta(\varepsilon)\right), \text{ for } t\ge t_0 \tag{19}$$

and hence by properties 2 and 3 we conclude

$$ {}_{t_0}^{c}D_t^q\left(V(t,x_t)+\alpha_3\left(\delta(\varepsilon)\right)\frac{(t-t_0)^q}{\Gamma(1+q)}\right)\le 0. \tag{20}$$

Then by using the property 4 we have

$$V(t,x_t)+\alpha_3(\delta(\varepsilon))\frac{(t-t_0)^q}{\Gamma(1+q)}\le V(t_0,\phi). \tag{21}$$

As a result we obtain

$$\begin{aligned} V(t,x_t) &\le V(t_0,\phi)-\alpha_3(\delta(\varepsilon))\frac{(t-t_0)^q}{\Gamma(1+q)} \\ &\le \alpha_2(\|\phi\|_c)-\alpha_3(\delta(\varepsilon))\frac{(t-t_0)^q}{\Gamma(1+q)} \\ &\le \alpha_2(\delta_0)-\alpha_3(\delta(\varepsilon))\frac{(t-t_0)^q}{\Gamma(1+q)}, \end{aligned} \tag{22}$$

which for $t=t_0+T(\varepsilon)$, reduces to

$$0<\alpha_1(\delta(\varepsilon))\le V\left(t_0+T,x_{t_0+T}\right)\le\alpha_2(\delta_0)-\frac{\alpha_3(\delta(\varepsilon))}{\Gamma(q+1)}T^q=0. \tag{23}$$

This contradiction proves that there exists a $t_1\in[t_0,t_0+T(\varepsilon)]$ such that $\|x(t_1)\|<\delta(\varepsilon)$. Thus, in any case we have $\|x(t)\|<\varepsilon$, $t\ge t_0+T(\varepsilon)$, whenever $\|x_{t_0}\|_c<\delta_0$, proving the uniform asymptotic stability of the trivial solution of (11).

Finally if $\lim_{s\to\infty}\alpha_1(s)=\infty$, then δ_0 above may be arbitrary large, and ε can be chosen after δ_0 is given to satisfy $\alpha_2(\delta_0)<\alpha_1(\varepsilon)$, and therefore global uniform asymptotic stability can be concluded.

We observe from the above proof that $\alpha_1,\alpha_2,\alpha_3$ and V(t,.) need only to be defined in a neighborhood of zero except for the case of global stability. We also notice that the lower bound of V need only to be a positive function of $\phi(0)$.

5. CONCLUSIONS

The combination of the fractional calculus and delay techniques seems to describe better the dynamics of the complex systems namely because both theories take into account the memory effects. In this paper we generalized the fractional Lyapunov-Krasovskii theorem in the presence of Caputo fractional derivatives and delay. The obtained theorem contains as particular cases the fractional calculus version as well as the time-delay one. The use of the Caputo fractional derivative was crucial for proving the obtained results.

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